Extra Questions for MA 109

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TA for D1-T3

Semester: Autumn 2020-21
Latest update: December 20, 2020

Notation:
$\mathbb{N} = \{1, 2, \ldots\}$ denotes the set of natural numbers.
$\mathbb{Z} = \mathbb{N} \cup \{0\} \cup \{-n : n \in \mathbb{N}\}$ denotes the set of integers.
$\mathbb{Q}$ denotes the set of rational numbers.
$\mathbb{R}$ denotes the set of real numbers.

§1. Sequences

1. Let (a_n) be a sequence of real numbers. We say that (a_n) is *slack-convergent* if there is an $a \in \mathbb{R}$ such that the following condition holds.
   For every $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $|a_n - a| \le \epsilon$ for all $n \ge n_0$.
   Prove or disprove that a sequence is convergent (in the normal sense) $\iff$ it is slack-convergent.

   (Additional) What happens if we change $n \ge n_0$ to $n > n_0$?

2. Let (a_n) be a sequence of real numbers. We say that (a_n) is *reciprocal-convergent* if there is an $a \in \mathbb{R}$ such that the following condition holds.
   For every $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $|a_n - a| < 1/\epsilon$ for all $n \ge n_0$.
   Prove or disprove that a sequence is convergent (in the normal sense) $\iff$ it is reciprocal-convergent.

3. Let (a_n) be a sequence of real numbers. We say that (a_n) is *natural-convergent* if the following condition holds.
   For every $k \in \mathbb{N}$, $\lim\limits_{n\to\infty} |a_{n+k} - a_n| = 0$.
   Prove or disprove that a sequence is convergent (in the normal sense) $\iff$ it is natural-convergent.

4. Let (a_n) be a sequence of real numbers. We say that (a_n) is *weirdly-convergent* if there is an $a \in \mathbb{R}$ such that the following condition holds.
   For every $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $|a_n - a| < \epsilon$ for infinitely many $n \ge n_0$.
   Prove or disprove that a sequence is convergent (in the normal sense) $\iff$ it is weirdly-convergent.

5. Let (a_n) be a sequence of real numbers. We say that (a_n) is *reverse-convergent* if there is an $a \in \mathbb{R}$ such that the following condition holds.
   For every $n_0 \in \mathbb{N}$, there is $\epsilon > 0$ such that $|a_n - a| < \epsilon$ for all $n \ge n_0$.
   Prove or disprove that a sequence is convergent (in the normal sense) $\iff$ it is reverse-convergent.

6. Let (a_n) be a bounded sequence such that
   $$\lim_{n\to\infty} (a_{n+1} - a_n)$$
   exists and equals 0.

   Prove/disprove: (a_n) is convergent.

7. Let f be any bijection from $\mathbb{N}$ to $\mathbb{Q} \cap [0, 1]$.
   Define the sequence (a_n) of real numbers as: $a_n := f(n) \quad \forall n \in \mathbb{N}$.
   Prove that (a_n) diverges or find an example of f such that (a_n) converges.

8. Let S be a nonempty subset of $\mathbb{R}$ which is bounded above. Let (a_n) be an increasing sequence in S such that $\lim_{n\to\infty} a_n = L \notin S$. (The sequence converges to a point outside S.)
   Prove or disprove that $L = \sup S$.

For the question(s) in which the implication does not hold in both directions, does it hold in any? If yes, which?

§2. Continuity

1. Show that $f : \mathbb{N} \to \mathbb{R}$ is continuous for any f.
2. Let $f : \mathbb{Q} \to \mathbb{R}$ be a continuous function such that the image (range) of f is a subset of $\mathbb{Q}$. Let $a,\ b,\ r \in \mathbb{Q}$ be such that $a < b$ and $f(a) < r < f(b)$. Show (with the help of an example) that it is not necessary that there exists some $c \in \mathbb{Q} \cap [a, b]$ such that $f(c) = r$.
3. (Dirichlet's function)
   Define $f : \mathbb{R} \to \mathbb{R}$ as
   $$f(x) := \begin{cases} 0 & x \in \mathbb{Q}, \\ 1 & x \notin \mathbb{Q}. \end{cases}$$
   Show that f is discontinuous everywhere.
4. (Thomae's function)
   Define $f : [0, 1] \to \mathbb{R}$ as follows:
   $$f(x) := \begin{cases} 0 & x \text{ is irrational,} \\ \dfrac{1}{n} & x = \dfrac{m}{n} \text{ in simplest form.} \end{cases}$$
   By "simplest form," we mean that $m \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$ with $\gcd(m, n) = 1$. (For 0, it will be $0/1$.)
   Show that f is discontinuous at all rationals in $[0, 1]$ and continuous at all other points in $[0, 1]$.
5. Let $f : \mathbb{R} \to \mathbb{R}$ and $c \in \mathbb{R}$. We say that f is *reverse continuous* at c if for all $\delta > 0$, there exists $\epsilon > 0$ such that $|x - c| < \delta \implies |f(x) - f(c)| < \epsilon$.
   Is this notion of continuity the same as the normal notion?
   If not, then give an example of a function which is reverse continuous at a point but not continuous or vice-versa.
6. Let $f : \mathbb{R} \to \mathbb{R}$ and $c \in \mathbb{R}$. We say that f is *upper continuous* at c if for all $\epsilon > 0$, there exists $\delta > 0$ such that $|x - c| < \delta \implies f(c) \le f(x) < f(c) + \epsilon$.
   (a) Prove that a function is continuous at a point if it is upper continuous at that point.
   (b) Show that the converse may not be true.
   (c) Give an example of a function that is upper continuous at only one point.
   (d) Given any $n \in \mathbb{N}$, show that there exists a function that is upper continuous at exactly n points.
   (e) Show that there exists a function that is upper continuous at infinitely many points.
   (f) Give an example of a function f that is upper continuous everywhere.
   (g) Can you give an example of another function g such that g is upper continuous everywhere but $f - g$ is not constant?
7. Let $A, B \subset \mathbb{R}$ and $f : A \to B$ be a bijection. Show with the help of an example that f is continuous $\not\Longrightarrow$ f^{-1} is continuous.
8. Show that there exists a bijection from $(0, 1)$ to $[0, 1]$.
9. Show that there exists no continuous bijection from $(0, 1)$ to $[0, 1]$ or from $[0, 1]$ to $(0, 1)$.
10. Let $f : A \to B$ be a continuous surjective function. Show that it is possible for A to be a bounded open interval and B to be a bounded closed interval.
    Is it possible for A to be a bounded closed interval and B to be a bounded open interval?
11. Let $f : \mathbb{R} \to \mathbb{R}$ be a function with the intermediate value property. Is it necessary that f is continuous *somewhere*?
12. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that given any $c \in \mathbb{R}$, the limit $\lim_{x\to c} f(x)$ exists. Is it necessary that f is continuous *somewhere*?

The last two questions are just for one to think about. I do not expect solutions for those.

§3. Differentiation

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Let $c \in \mathbb{R}$. Is it necessary that there exist $a,\ b \in \mathbb{R}$ such that $a < c < b$ and $f'(c) = \dfrac{f(b) - f(a)}{b - a}$?

2. Let $k \in \mathbb{N}$. Construct a function $f : \mathbb{R} \to \mathbb{R}$ that is k times differentiable everywhere but not $(k+1)$ times differentiable somewhere.

3. Construct a function $f : \mathbb{R} \to \mathbb{R}$ which is differentiable at only one point.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Suppose there is $\alpha \in \mathbb{R}$ such that for all $x \in \mathbb{R}$, $|f'(x)| \leq \alpha < 1$. Let $a_1 \in \mathbb{R}$ and set $a_{n+1} := f(a_n)$ for all $n \in \mathbb{N}$. Show that the sequence (a_n) converges.

5. Let $D \subset \mathbb{R}$. A function $f : D \to \mathbb{R}$ is said to be *convex* if
$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \quad \forall x,\ y \in D,\ \forall \lambda \in [0,1].$$
Prove that if I is an open interval and $f : I \to \mathbb{R}$ is convex, then f is continuous. Where did you use that I is an open interval?
Give an example to show that if J is not an open interval, then a convex function $f : J \to \mathbb{R}$ need not be continuous.

6. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a differentiable function. Show by example that $f'(x) = 0 \quad \forall x \in D$ does not imply that f is constant.

7. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a differentiable function.
We say that f is increasing if $\forall x,\ y \in D : x \leq y \implies f(x) \leq f(y)$.
Show by example that $f'(x) \geq 0 \quad \forall x \in D$ does not imply that f is increasing.

8. Show that the implication in the last two questions would be true if D were an interval.

9. Let A and B be open intervals in $\mathbb{R}$ and $f : A \to B$ be a bijection such that f is differentiable. Show that it is not necessary that f^{-1} is differentiable.

10. * Construct a function $f_1 : \mathbb{R} \to \mathbb{R}$ with the following properties or show that no such function exists:
1. f_1 is differentiable everywhere except one point x_1.
2. Define $f_2 : \mathbb{R} \setminus \{x_1\} \to \mathbb{R}$ as $f_2(x) :=$ derivative of f_1 at x. This f_2 must be differentiable everywhere in its domain except one point x_2.
3. Define $f_3 : \mathbb{R} \setminus \{x_1,\ x_2\} \to \mathbb{R}$ as $f_3(x) :=$ derivative of f_2 at x. This f_3 must be differentiable everywhere in its domain except one point x_3.
$\vdots$
n. Define $f_n : \mathbb{R} \setminus \{x_1, \cdots, x_{n-1}\} \to \mathbb{R}$ as $f_n(x) :=$ derivative of f_{n-1} at x. This f_n must be differentiable everywhere in its domain except one point x_n.
$\vdots$
(Note that we do not stop at any n.)

§4. Riemann integration

1. Define $f : [0,2] \to \mathbb{R}$ as
$$f(x) := \begin{cases} 0 & x \neq 1, \\ 1 & x = 1. \end{cases}$$
Show that f is Riemann integrable on $[0,2]$ and that the integral is 0.
Note that we had $f \geq 0$ with $\int_a^b f = 0$ and we still didn't get that f is identically zero.

2. Suppose $a < b$ and $f : [a,b] \to \mathbb{R}$ is continuous. Show that if $f \geq 0$ and $\int_a^b f = 0$, then $f \equiv 0$. That is, $f(x) = 0$ for all $x \in [a,b]$.
Compare this with the previous question.

3. Recall Dirichlet's function $f : \mathbb{R} \to \mathbb{R}$ from earlier:

$$f(x) := \begin{cases} 0 & x \in \mathbb{Q}, \\ 1 & x \notin \mathbb{Q}. \end{cases}$$

Show that f is not integrable over $[0, 1]$.

§5. General

Most of these questions are above the level of the course.

1. Let $D \subset \mathbb{R}$. We say a function $f : D \to \mathbb{R}$ is *uniformly continuous* if for all $\epsilon > 0$, there exists $\delta > 0$ such that whenever $x,\ y \in D$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.
   (a) Understand how this definition is different from the definition of (usual) continuity.
   (b) Give an example of a function which is continuous but not uniformly continuous.
   (c) Show that any uniformly continuous function is also continuous.
2. Let (f_n) be a sequence of real valued functions defined on $[a, b]$ such that each f_n is continuous. Moreover, you are given that for each $x \in [a, b]$, the limit $\lim\limits_{n\to\infty} f_n(x)$ exists.
   Define the function $f : [a, b] \to \mathbb{R}$ as follows:
   $$f(x) := \lim_{n\to\infty} f_n(x).$$
   Show with the help of an example that it is not necessary that f is continuous.
3. Let $f_n : D \to \mathbb{R}$ be a sequence of functions from the set $D \subset \mathbb{R}$ to $\mathbb{R}$. We say that the sequence (f_n) *converges uniformly* to the function $f : D \to \mathbb{R}$ if given $\epsilon > 0$, there exists an integer N such that
   $$|f_n(x) - f(x)| < \epsilon$$
   for all $n > N$ and all $x \in D$.
   Prove that if (f_n) is a sequence of continuous functions that converges uniformly to f, then f is continuous.
   If you have solved the previous question, show that (f_n) didn't uniformly converge to f for that example.
4. Let $f : [a, b] \to \mathbb{R}$ be any function. Then, we know that if
   (a) f is monotonic, or
   (b) f is bounded and has at most a finite number of discontinuities in $[a, b]$,

   then f is (Riemann) integrable.
   Is the converse true?
   That is, if f is (Riemann) integrable, then is it necessary that one of (a) or (b) should be true? Prove or disprove via counterexample. (Credit: Amit Kumar)
5. Show that any function $f : \mathbb{N} \to \mathbb{R}$ is uniformly continuous.
6. Let $a \in \mathbb{R}$ and (a_n) be a sequence of real numbers with the following property: Given any subsequence (a_{n_k}) of (a_n), there exists a subsequence $\left(a_{n_{k_l}}\right)$ of (a_{n_k}) with the property that $\lim\limits_{l\to\infty} a_{n_{k_l}} = a$.
   Prove that $\lim\limits_{n\to\infty} a_n = a$.
7. Let E be a bounded subset of $\mathbb{R}$ with the following property:
   There exists $x_0 \in \mathbb{R} \setminus E$ such that there exists a sequence (x_n) in E which converges to x_0. (For those familiar with the lingo, E is not a closed set.)
   Show that there exists:
   (a) A function $g : E \to \mathbb{R}$ which is continuous but not bounded.
   (b) A function $f : E \to \mathbb{R}$ such that $f(E)$ is bounded but does not have a maximum.
   (c) A function $h : E \to \mathbb{R}$ such that h is continuous but not uniformly continuous.

8. Let $f : (a, b) \to \mathbb{R}$ be a monotonically increasing function, that is, $a < x < y < b \implies f(x) \leq f(y)$.
   Show that for any $x \in (a, b)$, both $\lim\limits_{t \to x^-} f(t)$ and $\lim\limits_{t \to x^+} f(t)$ exist. Moreover, show that $\lim\limits_{t \to x^-} f(t) \leq f(x) \leq \lim\limits_{t \to x^+} f(t)$.

   Also show that if $x < y$, then $\lim\limits_{t \to x^+} f(t) \leq \lim\limits_{t \to y^-} f(t)$.

   (Hint: Try relating $\lim\limits_{t \to x^-} f(t)$ with $\sup\limits_{a<t<x} f(t)$.)

9. Let $S = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$. Show that given any $x \in \mathbb{R}$, there exists a sequence (s_n) in S that converges to x.
   Bonus 1: Generalise the argument by replacing $\sqrt{2}$ by any irrational square root of a natural number.
   Bonus 2: Generalise the argument by replacing $\sqrt{2}$ by any irrational number.

10. Let $f : \mathbb{R} \to \mathbb{R}$ be a periodic function with period $p > 0$. That is, $f(x + p) = f(x)$ for all $x \in \mathbb{R}$. Moreover, assume that f is Riemann integrable on $[x, x+p]$ for any $x \in \mathbb{R}$. Is it necessary that $\displaystyle\int_x^{x+p} f(x)dx$ is independent of x? (Note that f is not necessarily continuous.)

11. Let $A \subset \mathbb{R}$ and $f : A \to \mathbb{R}$ be a continuous and periodic function.
    (a) Show that if $A = \mathbb{R}$, then f is bounded.
    (b) Show that there exists some A and some f for which the hypothesis holds but f is not bounded.

12. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that it is differentiable at 0. Is it necessary that there exist $a < 0 < b$ such that f is continuous at every point in (a, b)?

13. Let $f : \mathbb{R} \to \mathbb{R}$ be a monotonically increasing function. Show that the set of discontinuities of f is countable.
    (A set E is said to be countable if there exists a one-to-one function from E to $\mathbb{N}$. Examples - $\varnothing$, $\{1, 5, 6\}$, $\mathbb{Q}$.)

14. Show with the help of an example that there exists a function $f : \mathbb{R} \to \mathbb{R}$ such that f is continuous and bounded but not uniformly continuous.

15. Suppose $E \subset \mathbb{R}$. Let $f : E \to \mathbb{R}$ be a uniformly continuous function. Show that if (x_n) is a convergent sequence in E, then the sequence $(f(x_n))$ converges in $\mathbb{R}$. (Hint: Cauchy)
    Show with the help of an example that the result need not hold if the function is just "continuous."

16. Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions such that $f(q) = g(q)$ for all $q \in \mathbb{Q}$.
    Show that $f(x) = g(x)$ for all $x \in \mathbb{R}$.
    Is the result true if we drop the continuity hypothesis?
    Can you think of a more general result? More simply, what sort of sets can we replace $\mathbb{Q}$ with?

17. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that f' is bounded. Show that f is uniformly continuous.

18. Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function. Suppose f has the property that $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. Is it necessary that there exists $\epsilon > 0$ such that f is constant in the interval $(-\epsilon, \epsilon)$?

19. Does there exist a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x)$ is rational iff x is irrational?