

Extra Questions for MA 109

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Notation:

$\mathbb{N} = \{1, 2, \dots\}$ denotes the set of natural numbers.

$\mathbb{Z} = \mathbb{N} \cup \{0\} \cup \{-n : n \in \mathbb{N}\}$ denotes the set of integers.

\mathbb{Q} denotes the set of rational numbers.

\mathbb{R} denotes the set of real numbers.

§1. Sequences

1. Let (a_n) be a sequence of real numbers. We say that (a_n) is *slack-convergent* if there is an $a \in \mathbb{R}$ such that the following condition holds.

For every $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $|a_n - a| \leq \epsilon$ for all $n \geq n_0$.

Prove or disprove that a sequence is convergent (in the normal sense) \iff it is slack-convergent.

(Additional) What happens if we change $n \geq n_0$ to $n > n_0$?

2. Let (a_n) be a sequence of real numbers. We say that (a_n) is *reciprocal-convergent* if there is an $a \in \mathbb{R}$ such that the following condition holds.

For every $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $|a_n - a| < 1/\epsilon$ for all $n \geq n_0$.

Prove or disprove that a sequence is convergent (in the normal sense) \iff it is reciprocal-convergent.

3. Let (a_n) be a sequence of real numbers. We say that (a_n) is *natural-convergent* if the following condition holds.

For every $k \in \mathbb{N}$, $\lim_{n \rightarrow \infty} |a_{n+k} - a_n| = 0$.

Prove or disprove that a sequence is convergent (in the normal sense) \iff it is natural-convergent.

4. Let (a_n) be a sequence of real numbers. We say that (a_n) is *weirdly-convergent* if there is an $a \in \mathbb{R}$ such that the following condition holds.

For every $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $|a_n - a| < \epsilon$ for infinitely many $n \geq n_0$.

Prove or disprove that a sequence is convergent (in the normal sense) \iff it is weirdly-convergent.

5. Let (a_n) be a sequence of real numbers. We say that (a_n) is *reverse-convergent* if there is an $a \in \mathbb{R}$ such that the following condition holds.

For every $n_0 \in \mathbb{N}$, there is $\epsilon > 0$ such that $|a_n - a| < \epsilon$ for all $n \geq n_0$.

Prove or disprove that a sequence is convergent (in the normal sense) \iff it is reverse-convergent.

6. Let (a_n) be a bounded sequence such that

$$\lim_{n \rightarrow \infty} (a_{n+1} - a_n)$$

exists and equals 0.

Prove/disprove: (a_n) is convergent.

7. Let f be any bijection from \mathbb{N} to $\mathbb{Q} \cap [0, 1]$.

Define the sequence (a_n) of real numbers as: $a_n := f(n) \quad \forall n \in \mathbb{N}$.

Prove that (a_n) diverges or find an example of f such that (a_n) converges.

8. Let S be a nonempty subset of \mathbb{R} which is bounded above. Let (a_n) be an increasing sequence in S such that $\lim_{n \rightarrow \infty} a_n = L \notin S$. (The sequence converges to a point outside S .)
 Prove or disprove that $L = \sup S$.

For the question(s) in which the implication does not hold in both directions, does it hold in any? If yes, which?

§2. Continuity

1. Show that $f : \mathbb{N} \rightarrow \mathbb{R}$ is continuous for any f .
2. Let $f : \mathbb{Q} \rightarrow \mathbb{R}$ be a continuous function such that the image (range) of f is a subset of \mathbb{Q} . Let $a, b, r \in \mathbb{Q}$ be such that $a < b$ and $f(a) < r < f(b)$. Show (with the help of an example) that it is not necessary that there exists some $c \in \mathbb{Q} \cap [a, b]$ such that $f(c) = r$.
3. (Dirichlet's function)
 Define $f : \mathbb{R} \rightarrow \mathbb{R}$ as

$$f(x) := \begin{cases} 0 & x \in \mathbb{Q}, \\ 1 & x \notin \mathbb{Q}. \end{cases}$$

Show that f is discontinuous everywhere.

4. (Thomae's function)
 Define $f : [0, 1] \rightarrow \mathbb{R}$ as follows:

$$f(x) := \begin{cases} 0 & x \text{ is irrational,} \\ \frac{1}{n} & x = \frac{m}{n} \text{ in simplest form.} \end{cases}$$

By "simplest form," we mean that $m \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$ with $\gcd(m, n) = 1$. (For 0, it will be $0/1$.)

Show that f is discontinuous at all rationals in $[0, 1]$ and continuous at all other points in $[0, 1]$.

5. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ and $c \in \mathbb{R}$. We say that f is *reverse continuous* at c if for all $\delta > 0$, there exists $\epsilon > 0$ such that $|x - c| < \delta \implies |f(x) - f(c)| < \epsilon$.
 Is this notion of continuity the same as the normal notion?
 If not, then give an example of a function which is reverse continuous at a point but not continuous or vice-versa.
6. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ and $c \in \mathbb{R}$. We say that f is *upper continuous* at c if for all $\epsilon > 0$, there exists $\delta > 0$ such that $|x - c| < \delta \implies f(c) \leq f(x) < f(c) + \epsilon$.
 - (a) Prove that a function is continuous at a point if it is upper continuous at that point.
 - (b) Show that the converse may not be true.
 - (c) Give an example of a function that is upper continuous at only one point.
 - (d) Given any $n \in \mathbb{N}$, show that there exists a function that is upper continuous at exactly n points.
 - (e) Show that there exists a function that is upper continuous at infinitely many points.
 - (f) Give an example of a function f that is upper continuous everywhere.
 - (g) Can you give an example of another function g such that g is upper continuous everywhere but $f - g$ is not constant?
7. Let $A, B \subset \mathbb{R}$ and $f : A \rightarrow B$ be a bijection. Show with the help of an example that f is continuous $\not\implies f^{-1}$ is continuous.
8. Show that there exists a bijection from $(0, 1)$ to $[0, 1]$.
9. Show that there exists no continuous bijection from $(0, 1)$ to $[0, 1]$ or from $[0, 1]$ to $(0, 1)$.
10. Let $f : A \rightarrow B$ be a continuous surjective function. Show that it is possible for A to be a bounded open interval and B to be a bounded closed interval.
 Is it possible for A to be a bounded closed interval and B to be a bounded open interval?
11. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function with the intermediate value property. Is it necessary that f is continuous *somewhere*?
12. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function such that given any $c \in \mathbb{R}$, the limit $\lim_{x \rightarrow c} f(x)$ exists. Is it necessary that f is continuous *somewhere*?

The last two questions are just for one to think about. I do not expect solutions for those.

§3. Differentiation

- Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function. Let $c \in \mathbb{R}$. Is it necessary that there exist $a, b \in \mathbb{R}$ such that $a < c < b$ and $f'(c) = \frac{f(b) - f(a)}{b - a}$?
- Let $k \in \mathbb{N}$. Construct a function $f : \mathbb{R} \rightarrow \mathbb{R}$ that is k times differentiable everywhere but not $(k + 1)$ times differentiable somewhere.
- Construct a function $f : \mathbb{R} \rightarrow \mathbb{R}$ which is differentiable at only one point.
- Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be differentiable. Suppose there is $\alpha \in \mathbb{R}$ such that for all $x \in \mathbb{R}$, $|f'(x)| \leq \alpha < 1$. Let $a_1 \in \mathbb{R}$ and set $a_{n+1} := f(a_n)$ for all $n \in \mathbb{N}$. Show that the sequence (a_n) converges.
- Let $D \subset \mathbb{R}$. A function $f : D \rightarrow \mathbb{R}$ is said to be *convex* if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) \quad \forall x, y \in D, \forall \lambda \in [0, 1].$$

Prove that if I is an open interval and $f : I \rightarrow \mathbb{R}$ is convex, then f is continuous. Where did you use that I is an open interval?

Give an example to show that if J is not an open interval, then a convex function $f : J \rightarrow \mathbb{R}$ need not be continuous.

- Let $D \subset \mathbb{R}$ and $f : D \rightarrow \mathbb{R}$ be a differentiable function. Show by example that $f'(x) = 0 \quad \forall x \in D$ does not imply that f is constant.
- Let $D \subset \mathbb{R}$ and $f : D \rightarrow \mathbb{R}$ be a differentiable function.
We say that f is increasing if $\forall x, y \in D : x \leq y \implies f(x) \leq f(y)$.
Show by example that $f'(x) \geq 0 \quad \forall x \in D$ does not imply that f is increasing.
- Show that the implication in the last two questions would be true if D were an interval.
- Let A and B be open intervals in \mathbb{R} and $f : A \rightarrow B$ be a bijection such that f is differentiable. Show that it is not necessary that f^{-1} is differentiable.
- * Construct a function $f_1 : \mathbb{R} \rightarrow \mathbb{R}$ with the following properties or show that no such function exists:
 - f_1 is differentiable everywhere except one point x_1 .
 - Define $f_2 : \mathbb{R} \setminus \{x_1\} \rightarrow \mathbb{R}$ as $f_2(x) :=$ derivative of f_1 at x . This f_2 must be differentiable everywhere in its domain except one point x_2 .
 - Define $f_3 : \mathbb{R} \setminus \{x_1, x_2\} \rightarrow \mathbb{R}$ as $f_3(x) :=$ derivative of f_2 at x . This f_3 must be differentiable everywhere in its domain except one point x_3 .
 - \vdots
 - Define $f_n : \mathbb{R} \setminus \{x_1, \dots, x_{n-1}\} \rightarrow \mathbb{R}$ as $f_n(x) :=$ derivative of f_{n-1} at x . This f_n must be differentiable everywhere in its domain except one point x_n .
 - \vdots
 (Note that we do not stop at any n .)

§4. Riemann integration

- Define $f : [0, 2] \rightarrow \mathbb{R}$ as

$$f(x) := \begin{cases} 0 & x \neq 1, \\ 1 & x = 1. \end{cases}$$

Show that f is Riemann integrable on $[0, 2]$ and that the integral is 0.

Note that we had $f \geq 0$ with $\int_a^b f = 0$ and we still didn't get that f is identically zero.

- Suppose $a < b$ and $f : [a, b] \rightarrow \mathbb{R}$ is continuous. Show that if $f \geq 0$ and $\int_a^b f = 0$, then $f \equiv 0$. That is, $f(x) = 0$ for all $x \in [a, b]$.
Compare this with the previous question.

3. Recall Dirichlet's function $f : \mathbb{R} \rightarrow \mathbb{R}$ from earlier:

$$f(x) := \begin{cases} 0 & x \in \mathbb{Q}, \\ 1 & x \notin \mathbb{Q}. \end{cases}$$

Show that f is not integrable over $[0, 1]$.

§5. General

Most of these questions are above the level of the course.

- Let $D \subset \mathbb{R}$. We say a function $f : D \rightarrow \mathbb{R}$ is *uniformly continuous* if for all $\epsilon > 0$, there exists $\delta > 0$ such that whenever $x, y \in D$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.
 - Understand how this definition is different from the definition of (usual) continuity.
 - Give an example of a function which is continuous but not uniformly continuous.
 - Show that any uniformly continuous function is also continuous.

- Let (f_n) be a sequence of real valued functions defined on $[a, b]$ such that each f_n is continuous. Moreover, you are given that for each $x \in [a, b]$, the limit $\lim_{n \rightarrow \infty} f_n(x)$ exists.

Define the function $f : [a, b] \rightarrow \mathbb{R}$ as follows:

$$f(x) := \lim_{n \rightarrow \infty} f_n(x).$$

Show with the help of an example that it is not necessary that f is continuous.

- Let $f_n : D \rightarrow \mathbb{R}$ be a sequence of functions from the set $D \subset \mathbb{R}$ to \mathbb{R} . We say that the sequence (f_n) *converges uniformly* to the function $f : D \rightarrow \mathbb{R}$ if given $\epsilon > 0$, there exists an integer N such that

$$|f_n(x) - f(x)| < \epsilon$$

for all $n > N$ and all $x \in D$.

Prove that if (f_n) is a sequence of continuous functions that converges uniformly to f , then f is continuous. If you have solved the previous question, show that (f_n) didn't uniformly converge to f for that example.

- Let $f : [a, b] \rightarrow \mathbb{R}$ be any function. Then, we know that if

- f is monotonic, or
- f is bounded and has at most a finite number of discontinuities in $[a, b]$,

then f is (Riemann) integrable.

Is the converse true?

That is, if f is (Riemann) integrable, then is it necessary that one of (a) or (b) should be true? Prove or disprove via counterexample. (Credit: Amit Kumar)

- Show that any function $f : \mathbb{N} \rightarrow \mathbb{R}$ is uniformly continuous.
- Let $a \in \mathbb{R}$ and (a_n) be a sequence of real numbers with the following property: Given any subsequence (a_{n_k}) of (a_n) , there exists a subsequence $(a_{n_{k_l}})$ of (a_{n_k}) with the property that $\lim_{l \rightarrow \infty} a_{n_{k_l}} = a$. Prove that $\lim_{n \rightarrow \infty} a_n = a$.

- Let E be a bounded subset of \mathbb{R} with the following property: There exists $x_0 \in \mathbb{R} \setminus E$ such that there exists a sequence (x_n) in E which converges to x_0 . (For those familiar with the lingo, E is not a closed set.) Show that there exists:
 - A function $g : E \rightarrow \mathbb{R}$ which is continuous but not bounded.
 - A function $f : E \rightarrow \mathbb{R}$ such that $f(E)$ is bounded but does not have a maximum.
 - A function $h : E \rightarrow \mathbb{R}$ such that h is continuous but not uniformly continuous.

8. Let $f : (a, b) \rightarrow \mathbb{R}$ be a monotonically increasing function, that is, $a < x < y < b \implies f(x) \leq f(y)$. Show that for any $x \in (a, b)$, both $\lim_{t \rightarrow x^-} f(t)$ and $\lim_{t \rightarrow x^+} f(t)$ exist. Moreover, show that $\lim_{t \rightarrow x^-} f(t) \leq f(x) \leq \lim_{t \rightarrow x^+} f(t)$.
- Also show that if $x < y$, then $\lim_{t \rightarrow x^+} f(t) \leq \lim_{t \rightarrow y^-} f(t)$.
- (Hint: Try relating $\lim_{t \rightarrow x^-} f(t)$ with $\sup_{a < t < x} f(t)$.)
9. Let $S = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$. Show that given any $x \in \mathbb{R}$, there exists a sequence (s_n) in S that converges to x .
- Bonus 1: Generalise the argument by replacing $\sqrt{2}$ by any irrational square root of a natural number.
 Bonus 2: Generalise the argument by replacing $\sqrt{2}$ by any irrational number.
10. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a periodic function with period $p > 0$. That is, $f(x + p) = f(x)$ for all $x \in \mathbb{R}$. Moreover, assume that f is Riemann integrable on $[x, x + p]$ for any $x \in \mathbb{R}$. Is it necessary that $\int_x^{x+p} f(x) dx$ is independent of x ? (Note that f is not necessarily continuous.)
11. Let $A \subset \mathbb{R}$ and $f : A \rightarrow \mathbb{R}$ be a continuous and periodic function.
- (a) Show that if $A = \mathbb{R}$, then f is bounded.
 (b) Show that there exists some A and some f for which the hypothesis holds but f is not bounded.
12. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function such that it is differentiable at 0. Is it necessary that there exist $a < 0 < b$ such that f is continuous at every point in (a, b) ?
13. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a monotonically increasing function. Show that the set of discontinuities of f is countable. (A set E is said to be countable if there exists a one-to-one function from E to \mathbb{N} . Examples - \emptyset , $\{1, 5, 6\}$, \mathbb{Q} .)
14. Show with the help of an example that there exists a function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that f is continuous and bounded but not uniformly continuous.
15. Suppose $E \subset \mathbb{R}$. Let $f : E \rightarrow \mathbb{R}$ be a uniformly continuous function. Show that if (x_n) is a convergent sequence in E , then the sequence $(f(x_n))$ converges in \mathbb{R} . (Hint: Cauchy)
 Show with the help of an example that the result need not hold if the function is just "continuous."
16. Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ be continuous functions such that $f(q) = g(q)$ for all $q \in \mathbb{Q}$. Show that $f(x) = g(x)$ for all $x \in \mathbb{R}$.
 Is the result true if we drop the continuity hypothesis?
 Can you think of a more general result? More simply, what sort of sets can we replace \mathbb{Q} with?
17. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function such that f' is bounded. Show that f is uniformly continuous.
18. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be an infinitely differentiable function. Suppose f has the property that $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. Is it necessary that there exists $\epsilon > 0$ such that f is constant in the interval $(-\epsilon, \epsilon)$?
19. Does there exist a continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(x)$ is rational iff x is irrational?