MA 109: Calculus I

Assignment Solutions

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Contents

§1. Assignment 1

Take the last digit of your roll number. Call it w. Take the second last digit of your roll number. Call it z. Let $a = w + 10$ and $b = z + 10$. Evaluate the limit

$$\lim_{n\to\infty} \frac{an+1}{bn+2}.$$

Justify your answer using the $\epsilon - N$ definition of the limit.

Solution. Calculate a and b first, as per **your** roll number (and not the roll number of the person you're copying from).

Claim. We claim that $\lim_{n\to\infty} \frac{an+1}{bn+2} = \frac{a}{b}$.

[You don't have to justify how you came up with the limit.]

We need to show that given *any* $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\left|\frac{an+1}{bn+2} - \frac{a}{b}\right| < \epsilon \quad \text{for all } n > N.$$

[Note the order of ϵ, N, and n. Changing this will change the meaning and won't be correct. Also note the "for all" in the two places.]

To this end, given any $\epsilon > 0$ define $N := \left\lfloor \frac{2a-b}{b^2\epsilon} \right\rfloor + 1$.

Now, note that if $n > N$, then

$$\begin{aligned}\left|\frac{an+1}{bn+2} - \frac{a}{b}\right| &= \frac{2a-b}{b(bn+2)} \qquad (\because b - 2a < 0)\\ &< \frac{2a-b}{b^2 n}\\ &< \frac{2a-b}{b^2 N}\\ &< \epsilon,\end{aligned}$$

as desired. □

An alternate:

Solution. Calculate a and b first, as per **your** roll number (and not the roll number of the person you're copying from).

Claim. We claim that $\lim\limits_{n\to\infty} \frac{an+1}{bn+2} = \frac{a}{b}$.

[You don't have to justify how you came up with the limit.]

We need to show that given *any* $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\left|\frac{an+1}{bn+2} - \frac{a}{b}\right| < \epsilon \quad \text{for all } n > N.$$

[Note the order of ϵ, N, and n. Changing this will change the meaning and won't be correct. Also note the "for all" in the two places.]

To this end, given any $\epsilon > 0$ and $n \in \mathbb{N}$, we note that

$$\begin{aligned}
\left|\frac{an+1}{bn+2} - \frac{a}{b}\right| < \epsilon &\iff \left|\frac{b-2a}{b(bn+2)}\right| < \epsilon \\
&\iff \frac{2a-b}{b(bn+2)} < \epsilon \\
&\Longleftarrow \frac{2a-b}{b^2 n} < \epsilon
\end{aligned}$$

$b - 2a < 0$ for your roll number

since $n \in \mathbb{N}$ and hence, $b(bn+2) > b(bn)$

Now, given any $\epsilon > 0$, we choose $N \in \mathbb{N}$ such that

$$N > \frac{2a-b}{b^2\epsilon}.$$

[You can also be explicit and choose $N = \left\lfloor \frac{2a-b}{b^2\epsilon} \right\rfloor + 1$ but make sure your quantity is then actually a positive integer.]

Thus, for $n > N$, we have

$$\frac{2a-b}{b^2 n} < \epsilon$$

and thus, by our earlier observation, we see

$$\left|\frac{an+1}{bn+2} - \frac{a}{b}\right| < \epsilon$$

for all $n > N$, as desired. □

§§1.1. Common mistakes

1. Not exactly a mistake but many of you spent a page or two in first "finding" the limit using Sandwich theorem and/or using that $1/n \to 0$. This is unnecessary.

2. If you have an inequality like $A - C < B - C$, you cannot (in general) conclude that $|A - C| < |B - C|$. In fact, in this case, $A - C$ was usually negative and thus, you need to justify more.

3. The direction of implication signs was in the opposite direction for many. (Note the red implication sign in the second solution. That's how it should be.) If you write something like $|a_n - l| < \epsilon \implies n > 1/\epsilon$, then simply choosing $N > 1/\epsilon$ does not solve the problem because you haven't said that $n > 1/\epsilon \implies |a_n - l| < \epsilon$.

4. **Don't** write something like $N = \dfrac{2a - b}{b^2\epsilon}$. You need N to be a positive integer.

5. Make sure you mention $\epsilon > 0$.

6. Note the definition says for all $\epsilon > 0$. You cannot choose ϵ on your own if you want to *prove* a limit. Something like "Set $\epsilon = \ldots$" is incorrect.

7. In the same vein as above, after you've fixed an $N \in \mathbb{N}$, the $|a_n - l| < \epsilon$ condition should hold for all $n > N$ and not *some*.

8. Speaking of fixing an N, do fix an N![1] Many of you have not done that.

9. Also, don't mess up the order of "for every $\epsilon > 0$, there exists $N \in \mathbb{N}$" by writing "there exists $N \in \mathbb{N}$ such that for every $\epsilon > 0$." The latter implies that a *single* $N \in \mathbb{N}$ works for *all* $\epsilon > 0$. That is *not* the case here.

10. Instead of writing "Assume that the limit is ...," you should write "We claim that the limit is ..." because you immediately follow it up with a proof. (The "assuming" thing is more appropriate when you want to disprove something by using contradiction.)

[1] Exclamation, not factorial.

§2. Assignment 2

Do there exist functions with the following properties? Justify.

1. $f : [0,1] \to \mathbb{R}$: convex and differentiable, $f'\left(\frac{1}{4}\right) = 2$, $f'\left(\frac{3}{4}\right) = -1$.
2. $f : [0,1] \to \mathbb{R}$: concave and discontinuous at $\frac{1}{2}$.
3. $f : [0,1] \to \mathbb{R}$: convex and differentiable and such that f' is not differentiable at $\frac{1}{2}$.
4. $f : \mathbb{R} \to \mathbb{R}$: strictly convex and strictly decreasing.

Solution. Note that if your answer is "Yes," then you must give an example and justify why it has the given properties. Otherwise, give a proof that no such function can exist.

1. No.

   Suppose not. Let f be a function with the given properties.

   Since f is convex and differentiable, f' must be increasing.

   Since $\frac{1}{4} \le \frac{3}{4}$, we must have $f'\left(\frac{1}{4}\right) \le f'\left(\frac{3}{4}\right)$. Since

   $$f'\left(\frac{3}{4}\right) = -1 < 2 = f'\left(\frac{1}{4}\right),$$

   we get a contradiction.

2. No.

   If a function $f : [a,b] \to \mathbb{R}$ is convex/concave, then it is continuous on (a,b). Here, we have $a = 0$ and $b = 1$. Since $\frac{1}{2} \in (0,1)$, we see that no such function is possible.

   [Note that this isn't given very precisely in slides. It is not true that f will be continuous on $[0,1]$. It may very well be discontinuous at the end-points. However, this time, no marks were deducted even if you wrote that "Convex/cave functions are continuous."]

3. Yes.

   Consider $f : [0,1] \to \mathbb{R}$ defined as

   $$f(x) := \begin{cases} 0 & 2x \le 1, \\ \left(x - \frac{1}{2}\right)^2 & 2x > 1. \end{cases}$$

Differentiability of f at any point in $[0,1] \setminus \left\{\frac{1}{2}\right\}$ is clear. At $1/2$, we compute the LHD and RHD as follows:

$$\begin{aligned}
\lim_{h \to 0^+} \frac{f\left(\frac{1}{2}+h\right) - f\left(\frac{1}{2}\right)}{h} &= \lim_{h \to 0^+} \frac{f\left(\frac{1}{2}+h\right) - 0}{h} \\
&= \lim_{h \to 0^+} \frac{\left(\dfrac{1}{2} + h - \dfrac{1}{2}\right)^2 - 0}{h} \\
&= \lim_{h \to 0^+} \frac{h^2}{h} = 0,
\end{aligned}$$

$$\begin{aligned}
\lim_{h \to 0^-} \frac{f\left(\frac{1}{2}+h\right) - f\left(\frac{1}{2}\right)}{h} &= \lim_{h \to 0^-} \frac{f\left(\frac{1}{2}+h\right) - 0}{h} \\
&= \lim_{h \to 0^-} \frac{0 - 0}{h} \\
&= 0.
\end{aligned}$$

Thus, we see that $f'\left(\frac{1}{2}\right) = 0$. For the other points, we use the usual formulae to get

$$f'(x) := \begin{cases} 0 & 2x \le 1, \\ 2\left(x - \dfrac{1}{2}\right) & 2x > 1. \end{cases}$$

To see that f' is not differentiable at $\frac{1}{2}$, we may compute the LHD and RHD again. One sees that the LHD is 0 whereas the RHD is 2.

Moreover, note that f' above is increasing, this tells us that f is convex.

4. Yes.

Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) := \exp(-x)$. Note that f is twice differentiable and that the following holds for all $x \in \mathbb{R}$:

$$\begin{aligned}
f'(x) &= -\exp(-x) < 0, \\
f''(x) &= \exp(-x) > 0.
\end{aligned}$$

The first inequality tells us that f is strictly decreasing while the second tells us that f is strictly convex. $\square$

§§2.1. Common mistakes

General

i. If your answer is "Yes," you cannot conclude by simply saying that "The definition of foo does not prevent bar and hence, it is possible." You must explicitly construct a example.

ii. You must justify your example in the above case. You cannot just state the function without explanation.

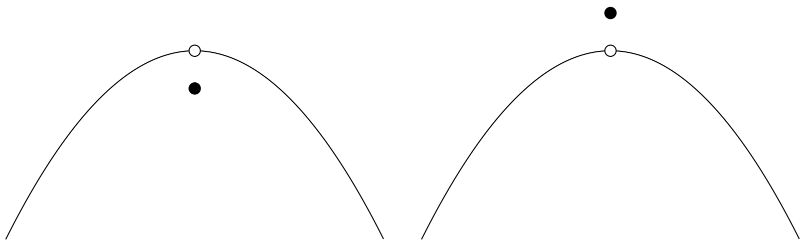
iii. You cannot simply draw a graph and justify your answer. Especially if your answer is "No."

Specific

Q1. You can't conclude by invoking f''. You don't know if it exists.

Q2.

i. Some of you tried drawing graphs like the following to disprove it.



Firstly, graphs are not proofs. Even ignoring that, all you have really shown is that this particular discontinuous function isn't concave. That doesn't prove anything.

ii. Some of you have tried arguing using:

$$\text{Concavity} \implies \text{Double derivative exists} \implies \text{Function is continuous}$$

This is not correct since concave functions don't even have to be differentiable. Consider $x \mapsto -|x|$ defined on $[-1, 1]$.

Q3.

i. This has got to be my favourite mistake I've (possibly ever) seen being made. Some of you have defined the function "piecewise" as:

$$f(x) = \begin{cases} x^2 + \frac{1}{4} & x \neq \frac{1}{2}, \\ \frac{1}{2} & x = \frac{1}{2} \end{cases}$$

and then proceeded to **incorrectly** differentiate it piecewise as

$$f'(x) = \begin{cases} 2x & x \neq \frac{1}{2}, \\ 0 & x = \frac{1}{2}. \end{cases}$$

The above is **absurd**.

Firstly, note that the function defined above is actually just $f(x) = x^2 + \frac{1}{4}$ and as such, is infinitely differentiable everywhere.

Secondly, the reason you cannot do that is because for you to differentiate "piecewise," you need an interval around that point on which the function doesn't change definition. This is something I had mentioned in Tutorial 2.

ii. In the same vein as above, you need to explicitly compute the first (and lack thereof of second) derivative at $\frac{1}{2}$ using the first principle (i.e., LHD and RHD). You cannot differentiate piecewise and argue using limits without additional justification.

iii. Recall the slide in recap where I had put a "please." I had requested you to remember that a function can be differentiable at a point without the derivative being continuous at that point. As such, all arguments using just limits of f' (or f'') have been discarded. (You could possibly argue using that but that needs more justification such as Darboux's theorem or something else.)

iv. Some of you had written something like

> $$f''(x) := \begin{cases} 4 & x \geq \frac{1}{2}, \\ 2 & x < \frac{1}{2} \end{cases}$$
>
> and hence, $f''\left(\frac{1}{2}\right)$ does not exist because the left and right hand limits of f'' don't agree.

This is **absurd**. To begin with, refer to the previous point.

Even keeping that aside: You, technically, are already saying $f''\left(\frac{1}{2}\right) = 4$ in the above and hence, according to *you*, f' is indeed differentiable at $\frac{1}{2}$.

v. Some of you have tried to conclude convexity by saying that $f''(x) \geq 0$. Firstly, f'' doesn't even exist at $\frac{1}{2}$. Secondly, if you are arguing by saying that "The second derivative is positive, wherever it exists," that is also not correct. For example, consider $g : \mathbb{R} \to \mathbb{R}$ defined as

$$g(x) := \begin{cases} (x+1)^2 - 1 & x < 0, \\ x^2 & x \geq 0. \end{cases}$$

g'' is defined on $\mathbb{R} \setminus \{0\}$ and is positive there. However, g is not convex.

Q4. This was fine for most of you. Do justify why f has the properties using f' and f'' (if you pick a twice differentiable function, like I did). Don't draw graphs.

§3. Assignment 3

Q1. Do there exist differentiable functions with the following properties?

1. $f : [0, 1] \to \mathbb{R}$, $f(0) = -2$, $f(1) = 3$, $f'(x) \geq 10$ for all $x \in (0, 1)$.
2. $f : [0, 1] \to \mathbb{R}$, $|f(x)| \leq 1$ for all $x \in (0, 1)$, and $\exists x \in (0, 1) : f'(x) \geq 2$.
3. $f : [0, 1] \to \mathbb{R}$, $f(0) = f(1)$, and $f(x) \neq f\left(x + \frac{1}{3}\right)$ for all $x \in \left(0, \frac{2}{3}\right)$.
4. $f : [0, 1] \to \mathbb{R}$, $f(0) = f(1)$, and $f(x) \neq f\left(x + \frac{3}{4}\right)$ for all $x \in \left(0, \frac{1}{4}\right)$.

Solution. The convention that I will adopt is that the function is differentiable on $(0, 1)$ and continuous on $[0, 1]$. We shall freely use the fact that differentiable functions are continuous.

1. No.

   Suppose not. Let f be a function with those properties.

   By the above written convention, we may apply MVT and deduce the existence of some $c \in (0, 1)$ such that

   $$f'(c) = \frac{f(1) - f(0)}{1 - 0} = 5 \ngeq 10.$$

   A contradiction.

2. Yes.

   Consider $f : [0, 1] \to \mathbb{R}$ defined as

   $$f(x) := 2x - 1.$$

   The above function is differentiable, being a polynomial.

   Note that

   $$x \in [0, 1] \implies 2x \in [0, 2] \implies 2x - 1 \in [-1, 1] \implies |f(x)| \leq 1.$$

   On the other hand, $f'\left(\frac{1}{2}\right) = 2 \geq 2$.

   Thus, f satisfies all the properties.

3. No.

   Suppose not. Let f be a function with those properties.

Consider $g : \left[0, \frac{2}{3}\right] \to \mathbb{R}$ defined by

$$g(x) := f(x) - f\left(x + \frac{1}{3}\right).$$

It suffices to show that g has a root in the open interval $\left(0, \frac{2}{3}\right)$.

Using the fact that $f(0) = f(1)$, we see that

$$g(0) + g\left(\frac{1}{3}\right) + g\left(\frac{2}{3}\right) = 0.$$

If $g\left(\frac{1}{3}\right) = 0$, then we are done. If that is not the case, then one of $g(0)$ or $g\left(\frac{2}{3}\right)$ must have sign opposite to that of $g\left(\frac{1}{3}\right)$. By IVT, g is zero at some point strictly between the two. (g is continuous since f is.) Thus, we are done.

4. Yes.

Consider $f : [0, 1] \to \mathbb{R}$ defined as $f(x) := \sin(2\pi x)$.

Clearly, f is differentiable and $f(0) = 0 = f(1)$. Moreover, f is strictly positive on $\left(0, \frac{1}{2}\right)$ and strictly negative on $\left(\frac{1}{2}, 1\right)$.

Thus, if $x \in \left(0, \frac{1}{4}\right) \subset \left(0, \frac{1}{2}\right)$, then $f(x) > 0$.
On the other hand, $x + \frac{3}{4} \in \left(\frac{3}{4}, 1\right) \subset \left(\frac{1}{2}, 1\right)$ and hence

$$f\left(x + \frac{3}{4}\right) < 0 < f(x).$$

Thus, we are done. □

Q2. Let $f : [0, 1) \to \mathbb{R}$ be defined as $f(x) = \log(1 + x)$. Let P_n denote the order n Taylor polynomial at the point $x_0 = 0$. Use Taylor's Theorem to find the smallest n so that the remainder term $R_n(x) = f(x) - P_n(x)$ satisfies

$$|R_n(x)| < 0.01$$

for all $x \in [0, 1)$.

Solution. For the moment, we consider our domain to be $[0, 1]$. (If you're not comfortable with derivatives at end points, consider the domain to be $(-1/2, 3/2)$ with $f(x) := \log(1 + x)$.)

First, note that

$$f^{(n+1)}(x) = \frac{(-1)^n n!}{(1 + x)^{n+1}}$$

for all $x \in [0, 1]$ and $n \geq 0$. Thus, we get the Taylor polynomials as

$$P_n(x) = -\sum_{k=1}^{n} \frac{(-1)^k x^k}{k}.$$

Moreover, by Taylor's Theorem, we see that

$$|R_n(x)| = \left| f^{(n+1)}(c_x) \frac{x^{n+1}}{(n + 1)!} \right| = \frac{x^{n+1}}{(1 + c_x)^{n+1}} \frac{1}{n + 1} \leq \frac{1}{n + 1}.$$

The last inequality follows since $0 \leq c_x < x \leq 1$.

Thus, for all $x \in [0, 1]$, we see that $P_n(x) \to f(x)$. In particular, we get the following.

Theorem 1

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

We note the following consequences.

Corollary 2

If $n \geq 1$, the following series converges:

$$\frac{1}{n} - \frac{1}{n + 1} + \frac{1}{n + 2} - \cdots .$$

Moreover, the above series has a strictly positive sum.

II

The convergence is clear since removing the finitely many terms from the beginning does not affect convergence. To see the positive sum, note that the subsequence of even partial sums is strictly increasing and strictly positive.

Now, note that using our formula for P_n, we get

$$R_n(x) = \log(1+x) + \sum_{k=1}^{n} \frac{(-1)^k x^k}{k}$$

for all $x \in [0, 1]$. (Note that since we're taking only finite sum on the right, the above is true even if you extended the domain to the open interval.)

Differentiating gives

$$\begin{aligned} R_n'(x) &= \frac{1}{1+x} + \sum_{k=1}^{n} (-1)^k x^{k-1} \\ &= \frac{1}{1+x} + \frac{(-1)(1-(-x)^n)}{1+x} \\ &= (-1)^n \frac{x^n}{1+x}. \end{aligned}$$

Thus, for all $x \in [0, 1] : R_n'(x) \le 0$ if n is odd and $R_n'(x) \ge 0$ if n is even. Furthermore, the inequalities are strict if $x \neq 0$.

Note that in either case, $R_n(0) = 0$. Thus, we see that $|R_n|$ is an increasing function on $[0, 1]$ and achieves its maximum at 1. In other words,

$$|R_n(x)| < |R_n(1)|$$

for all $x \in [0, 1)$.

Our aim now is to bound $|R_n(1)|$. For easier notation, let $e_n := |R_n(1)|$.

Using the corollary about the sign of the partial sums, we see that

$$\begin{aligned} e_n &= \frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+3} - \frac{1}{n+4} + \cdots, \\ e_{n+1} &= \frac{1}{n+2} - \frac{1}{n+3} + \frac{1}{n+4} - \frac{1}{n+5} + \cdots. \end{aligned}$$

Again, by comparing the even subsequences of partial sums, we see that

$$e_n > e_{n+1}$$

or

$$2e_n > e_n + e_{n+1}. \tag{$*$}$$

Looking at the explicit series for the quantities for the right, we see that $e_n + e_{n+1} = \frac{1}{n+1}$. Thus,

$$e_n > \frac{1}{2(n+1)}.$$

Similar to ($*$), we get the other inequality

$$2e_n < e_n + e_{n-1} = \frac{1}{n}.$$

Thus, we have the bound

$$\frac{1}{2(n+1)} < |R_n(1)| < \frac{1}{2n}.$$

Note that $1/0.01 = 100$ and thus, we see that if $n \leq 49$, then

$$R_n(1) > \frac{1}{2(n+1)} \geq \frac{1}{2 \cdot 50} = 0.01$$

and hence, $n \geq 50$. (Note that if $R_n(1) > 0.01$, then continuity of R_n implies that $R_n(x) > 0.01$ for some $x \in [0, 1)$.)

On the other hand, note that if $n = 50$, then

$$|R_n(1)| < \frac{1}{102} < \frac{1}{100} = 0.01.$$

Thus, $n = 50$. □

Using the above method, we get the following answers.

Error	n
0.01	50
0.02	25
0.03	16 or 17
0.04	12 or 13
0.05	10

By the "or," I mean that my method above does not give a conclusive answer. At this point, one may use a calculator and see that 17 works for 0.03 and 12 for 0.04.

§§3.1. Common mistakes

Q1. 3. A lot of you have tried to use some "reverse" Rolle's theorem. Note that if $f'(c) = 0$ for some c, that does not mean that you can find $a < c < b$ such that $f(a) = f(b)$.

Of course, any argument which can also work for the next part is clearly wrong.

Arguments using graphs are not correct.