# Overview

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$f: (0, \infty) \to \mathbb{R}$

$\mathcal{L}(f)(s) := \int_0^\infty f(t)e^{-st}\,dt$

$f \rightsquigarrow F$
$x \rightsquigarrow X$
$y \rightsquigarrow Y$

**Theorem 9.3: Derivatives of Laplace** $\mathcal{L}(f) = F$

$$\mathcal{L}(tf(t)) = -\frac{d}{ds}F(s).$$

In general,

$$\mathcal{L}(t^n f(t)) = (-1)^n \frac{d^n}{ds^n}F(s).$$

**Theorem 9.4: Laplace of derivatives** $\mathcal{L}(f) = F$

Most useful

$$\mathcal{L}(f'(t)) = sF(s) - f(0),$$
$$\mathcal{L}(f''(t)) = s^2F(s) - sf(0) - f'(0),$$
$$\mathcal{L}(f^{(n)}(t)) = s^nF(s) - \sum_{k=0}^{n-1} s^{n-1-k} f^{(k)}(0).$$

**Theorem 9.5: First Shift Theorem**

If

$$\mathcal{L}(f(t)) = F(s),$$

then

$$\mathcal{L}(e^{at}f(t)) = F(s-a).$$

$\mathcal{L}^{-1}(1/s) = 1$

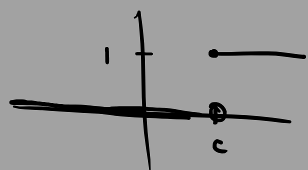
$\mathcal{L}^{-1}\left(\frac{1}{s-a}\right) = e^{at}$

| $f(t)$ | $F(s)$ | $f(t)$ | $F(s)$ |
|---|---|---|---|
| $t$ | $1/s^2$ | $t^a$ | $\dfrac{\Gamma(a+1)}{s^{a+1}}$ |
| $u_c(t)$ | $e^{-cs}/s$ | $e^{at}$ | $\dfrac{1}{s-a}$ |
| $\sin(\omega t)$ | $\dfrac{\omega}{s^2+\omega^2}$ | $\cos(\omega t)$ | $\dfrac{s}{s^2+\omega^2}$ |
| $t\sin(\omega t)$ | $\dfrac{2\omega s}{(s^2+\omega^2)^2}$ | $t\cos(\omega t)$ | $\dfrac{s^2-\omega^2}{(s^2+\omega^2)^2}$ |
| $e^{at}\sin(\omega t)$ | $\dfrac{\omega}{(s-a)^2+\omega^2}$ | $e^{at}\cos(\omega t)$ | $\dfrac{s-a}{(s-a)^2+\omega^2}$ |
| $\sinh(\omega t)$ | $\dfrac{\omega}{s^2-\omega^2}$ | $\cosh(\omega t)$ | $\dfrac{s}{s^2-\omega^2}$ |
| $e^{at}\sinh(\omega t)$ | $\dfrac{\omega}{(s-a)^2-\omega^2}$ | $e^{at}\cosh(\omega t)$ | $\dfrac{s-a}{(s-a)^2-\omega^2}$ |

**Definition 9.6: Heaviside step function**

The Heaviside unit step function $u : \mathbb{R} \to \{0, 1\}$ is defined as

$$u(t) := \begin{cases} 0 & \text{if } t < 0 \\ \underline{1} & \text{if } t \geq 0 \end{cases}$$




For $c \in \mathbb{R}$, the function $\underline{\underline{u_c(t)}}$ is defined as $u(t-c)$.

**Theorem 9.7: Second Shift Theorem**

Suppose $\mathcal{L}f = F(s)$ for $s > a \geq 0$.
If $c > 0$, then we have

$$\mathcal{L}(u_c(t)f(t-c)) = e^{-cs}F(s),$$

for $s > a$.

Given $f, g$, we define $f*g$ to be a new function given as

$$(f*g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau.$$

$$\mathcal{L}(f*g) = \mathcal{L}(f)\,\mathcal{L}(g).$$

$$f*g = g*f$$

$$f*(g+h) = f*g + f*h$$

$\vdots$

$f*1 \neq f$ in general

$\sin * 1 \neq \sin.$

Q: Does there exist some function $e$ such that $f*e = f$ for all (nice) functions $f$?

$$\frac{\overset{p(s)}{\text{polynomial in } s}}{(s-a)(s-b)(s-c)(s-d)} = \frac{A}{s-a} + \frac{B}{s-b} + \frac{C}{s-c} + \frac{D}{s-d} \quad (*)$$

for $A, B, C, D \in \mathbb{C}$.

$(a, b, c, d \in \mathbb{C}$ distinct$)$

How do we find $A$?

Multiply $(*)$ with $s-a$ and put $s=a$.

$$A = \frac{p(a)}{(a-b)(a-c)(a-d)}$$

# Q1. (v)

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**Q.1. Find the Laplace Transform of (v) $(1+te^{-t})^3$**

$$f(t) = (1+te^{-t})^3$$
$$= 1 + 3te^{-t} + 3t^2e^{-2t} + t^3e^{-3t}$$

Recall: $\mathcal{L}(t^a) = \dfrac{\Gamma(a+1)}{s^{a+1}}$

$\mathcal{L}(e^{bt}f(t)) = F(s-b)$

$\left(\Gamma(a+1) = a! \text{ for } a \in \mathbb{N} \cup \{0\}.\right)$

Combined: $\mathcal{L}(e^{bt}t^a) = \dfrac{\Gamma(a+1)}{(s-b)^{a+1}}$

$$\therefore \mathcal{L}(f(t)) = \mathcal{L}(1) + 3\mathcal{L}(te^{-t}) + 3\mathcal{L}(t^2e^{-2t}) + \mathcal{L}(t^3e^{3t})$$
$$= \frac{1}{s} + \frac{3}{(s+1)^2} + \frac{3\cdot 2}{(s+2)^3} + \frac{6}{(s+3)^4}$$
$$= \frac{1}{s} + \frac{3}{(s+1)^2} + \frac{6}{(s+2)^3} + \frac{6}{(s+3)^4}.$$

# Q2. (iv)

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Q.2. Find the inverse Laplace transforms of (iv) $\dfrac{s^3}{(s^4+4a^4)}$

(Sophie - Germain)

$$s^4 + 4a^4 = s^4 + 4a^2s^2 + 4a^4 - 4a^2s^2$$
$$= (s^2+2a^2)^2 - (2as)^2$$
$$= (s^2+2as+2a^2)(s^2-2as+2a^2)$$
$$= ((s+a)^2+a^2)((s-a)^2+a^2)$$
$$= (s+a+ia)(s+a-ia)(s-a+ia)(s-a-ia)$$

$$\frac{s^3}{(s^4+4a^4)} = \frac{A}{s+a+ia} + \frac{B}{s+a-ia} + \frac{C}{s-a+ia} + \frac{D}{s-a-ia}$$

(Multiply with $s+a+ia$ and put $s = -(a+ia)$ to get $A$.)

$$\frac{s^3(s+a+ia)}{s^4+4a^4} = A + (s+a+ia)\left(\frac{B}{\dots} + \frac{C}{\dots} + \frac{D}{\dots}\right)$$

$$\parallel$$

$$\frac{s^3}{(s+a-ia)(s-a+ia)(s-a-ia)}$$

Putting $s = -(a+ia)$, we get

$$A = \frac{(-1)^3 (a+ia)^{\cancel{3}\,2}}{(-2ia)(-2a)(-2)\cancel{(a+ia)}}$$

$$= \frac{1}{8(ia^2)}(a+ia)^2$$

$$= \frac{1}{8i}(1+i)^2 = \frac{2i}{8i} = \frac{1}{4}.$$

$$= \frac{1}{8i}(1+i)^2 = \frac{2i}{8i} = \frac{1}{4}.$$

Similarly, compute the others. You shall get
$B = C = D = 1/4.$

$$\frac{s^3}{s^4+4a^4} = \frac{1}{4}\left(\frac{1}{s+a+ia} + \frac{1}{s+a-ia} + \frac{1}{s-a+ia} + \frac{1}{s-a-ia}\right)$$

$$\left[\text{Recall: } \mathcal{L}^{-1}\left(\frac{1}{s-b}\right) = e^{bt}.\right]$$

$$\mathcal{L}^{-1}\left(\frac{s^3}{s^4+4a^4}\right) = \frac{1}{4}\left(e^{(a+ia)t} + e^{(a+ia)t} + e^{(-a+ia)t} + e^{(-a-ia)t}\right)$$

$$= \frac{1}{4}\left\{e^{at}\left(e^{iat}+e^{-iat}\right) + e^{-at}\left(e^{iat}+e^{-iat}\right)\right\}$$

$$= \left(\frac{e^{at}+e^{-at}}{2}\right)\left(\frac{e^{iat}+e^{-iat}}{2}\right)$$

$$= \cosh(at)\cos(at).$$

# Q3. (vi)

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Q.3. Solve the following intial value problems using Laplace transforms

$$y'' - 2y - 3y = 10\sinh 2t; \qquad y(0) = 0;\ y'(0) = 4$$

if $\mathcal{L}(y) = Y$, then

$$\mathcal{L}(y')(s) = sY(s) - y(0) = sY(s)$$

$$\mathcal{L}(y'')(s) = s^2 Y(s) - sy(0) - y'(0) = s^2Y(s) - 4$$

Take Laplace

$$[s^2Y(s) - 4] - 2[sY(s)] - 3Y(s) = (10)\frac{2}{s^2 - 2^2}$$

$$\Rightarrow (s^2 - 2s - 3)Y(s) - 4 = \frac{20}{s^2 - 4}$$

$$\Rightarrow Y(s) = \frac{1}{s^2 - 2s - 3}\left\{\frac{20}{s^2-4} + 4\right\}$$

$$= \frac{1}{s^2 - 2s - 3}\left\{\frac{4s^2 + 4}{s^2 - 4}\right\}$$

$$= \frac{4(s^2+1)}{(s+1)(s-3)(s+2)(s-2)}.$$

As before, we decompose into partial fractions.

$$(*)\quad \frac{4(s^2+1)}{(s+1)(s-3)(s+2)(s-2)} = \frac{A}{s+1} + \frac{B}{s-3} + \frac{C}{s+2} + \frac{D}{s-2}$$

$$A = \frac{4[(-1)^2+1]}{\ldots} = \frac{4(2)}{\ldots} = \frac{2}{3}$$

$$\overline{(-1-3)(-1+2)(-1-2)} = \overline{(-4)(1)(-3)} = \overline{3}$$

$$B = \frac{4\left[3^2+1\right]}{(3+1)(3+2)(3-2)} = \frac{4\cdot 10}{(4)(5)(1)} = 2$$

$$C = \frac{4\left[(-2)^2+1\right]}{(-2+1)(-2-3)(-2-2)} = \frac{4\cdot 5}{(-1)(-5)(-4)} = -1$$

$$D = \frac{4\left[2^2+1\right]}{(2+1)(2-3)(2+2)} = \frac{4\cdot 5}{(3)(-1)(4)} = -\frac{5}{3}$$

Taking $\mathcal{L}^{-1}(*)$ gives:

$$y(t) = Ae^{-t} + Be^{3t} + Ce^{-2t} + De^{2t}$$

$$\boxed{y(t) = \frac{2}{3}e^{-t} + 2e^{3t} - 1e^{-2t} - \frac{5}{3}e^{2t}.}$$

# Q4. (vi)

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Q.4. Solve the following systems of differential equations using Laplace transforms.

$$\begin{aligned} y_1'' + y_2 &= -5\cos 2t \\ y_1 + y_2'' &= 5\cos 2t \end{aligned} \quad \Bigg\| \quad \begin{aligned} &y_1(0) = 1,\ y_1'(0) = 1 \\ &y_2(0) = -1,\ y_2'(0) = 1 \end{aligned}$$

Taking Laplace transform gives:

$$\begin{cases} s^2 Y_1 - s y_1(0) - y_1'(0) + Y_2 = -5 \dfrac{s}{s^2+4} \\ Y_1 + s^2 Y_2 - s y_2(0) - y_2'(0) = \dfrac{5s}{s^2+4} \end{cases}$$

Plug in values.

$$s^2 Y_1 + Y_2 = \frac{-5s}{s^2+4} + s + 1$$

$$= \frac{-5s + s^3 + s^2 + 4s + 4}{s^2+4}$$

$$\boxed{s^2 Y_1 + Y_2 = \frac{s^3 + s^2 - s + 4}{s^2+4}} \quad \text{— (1)}$$

Similarly, the second equation becomes:

$$Y_1 + s^2 Y_2 = \frac{5s}{s^2+4} - s + 1$$

$$\boxed{Y_1 + s^2 Y_2 = \frac{-s^3 + s^2 + s + 4}{s^2+4}} \quad \text{— (2)}$$

Just add (1) and (2) :

$$(s^2+1)(Y_1 + Y_2) = \frac{2(s^2+4)}{s^2+4} = 2$$

$$\therefore \quad Y_1 + Y_2 = \frac{2}{s^2+1}$$

$\mathcal{L}^{-1}$

$$y_1(t) + y_2(t) = 2\sin(t) \quad \text{— (I)}$$

Similarly, subtracting (2) from (1) gives:

$$(s^2-1)(Y_1 - Y_2) = \frac{2(s^3 - s)}{s^2+4} = \frac{2s(s^2-1)}{s^2+4}$$

$$Y_1 - Y_2 = \frac{2s}{s^2+4}$$

$\mathcal{L}^{-1}$

$$y_1(t) - y_2(t) = 2\cos(2t) \quad \text{— (II)}$$

From (I) and (II), we get

$$y_1(t) = \sin(t) + \cos(2t), \quad \text{and}$$
$$y_2(t) = \sin(t) - \cos(2t).$$

Q5. (iv)

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$$\mathcal{L}^{-1}\left[\frac{?}{s} + \frac{?}{s^2} + \frac{?}{s^3} + \cdots\right] =$$

Q.5. Assuming that for a Power series in $\frac{1}{s}$ with no constant term the Laplace transform can be obtained term-by-term, i.e., assuming that $\left[\mathcal{L}^{-1}[\sum_0^\infty \frac{A_k}{s^{k+1}}] = \sum_0^\infty A_k \frac{t^k}{k!}\right]$, where $A_0, A_1 \ldots A_k \ldots$ are real numbers, prove that

$$\mathcal{L}^{-1}\left(\frac{1}{\sqrt{s^2+a^2}}\right) = \underbrace{J_0(at)}_{\text{Bessel function}}, \quad \text{for } a > 0;$$

where

$$J_0(t) := \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k}(k!)^2} t^{2k}.$$

$$\frac{1}{\sqrt{s^2+a^2}} = (s^2+a^2)^{-1/2} = \frac{1}{s}\left[(a/s)^2+1\right]^{-1/2}$$

$$= \frac{1}{s}\left[1 + \left(\frac{a}{s}\right)^2\right]^{-1/2}$$

$$= \frac{1}{s}\left\{1 + \frac{1}{1!}\left(-\frac{1}{2}\right)\left(\frac{a}{s}\right)^2 + \frac{1}{2!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{a}{s}\right)^4 + \cdots\right]$$

$$= \frac{1}{s}\sum_{k=0}^{\infty} \frac{1}{k!}(-1)^k \frac{1\cdot 3\cdot \cdots \cdot 2k-1}{2^k}\left(\frac{a}{s}\right)^{2k}$$

$$\left\{\begin{aligned} 1\cdot 3\cdot \cdots \cdot (2k-1) &= 1\cdot\frac{2}{2}\cdot 3\cdot\frac{4}{4}\cdot \cdots \cdot 2k-1\cdot\frac{2k}{2k} \\ &= \frac{(2k)!}{2\cdot 4\cdot \cdots \cdot 2k} \\ &= \frac{(2k)!}{2^k[1\cdot 2\cdot \cdots \cdot k]} = \frac{(2k)!}{2^k(k!)} \end{aligned}\right\}$$

$$= \frac{1}{s} \sum_{k=0}^{\infty} \frac{1}{k!} (-1)^k \frac{(2k)! / [2^k \cdot k!]}{2^k} \left(\frac{a}{s}\right)^{2k}$$

$$= \frac{1}{s} \sum_{k=0}^{\infty} (-1)^k \frac{(2k)!}{2^{2k} (k!)^2} \frac{a^{2k}}{s^{2k}}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k (2k)!}{2^{2k} (k!)^2} \cdot a^{2k} \cdot \frac{1}{s^{2k+1}}$$

Thus, taking term-by-term inverse Laplace, we get

$$\mathcal{L}^{-1}\left(\frac{1}{\sqrt{s^2+a^2}}\right) = \sum_{k=0}^{\infty} \left[\frac{(-1)^k (2k)!}{2^{2k} \cdot (k!)^2} \cdot a^{2k}\right] \frac{t^{2k}}{(2k)!}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k} \cdot (k!)^2} \cdot (at)^{2k}$$

$$= J_0(at).$$

$\blacksquare$

# Q10.

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$$\text{Q.10. Find } \mathcal{L}^{-1}\left[\ln\frac{s^2+4s+5}{s^2+2s+5}\right].$$

$$\mathcal{L}^{-1}\left\{\ln\left[\frac{s^2+4s+5}{s^2+2s+5}\right]\right\} = f(t)$$

$$\Rightarrow \mathcal{L}(f(t)) = \ln\left[\quad\right]$$

$$\Rightarrow \mathcal{L}(f(t)) = \ln(s^2+4s+5) - \ln(s^2+2s+5)$$

"ingenious"

$$\Rightarrow \mathcal{L}(t\,f(t)) = -\frac{d}{ds}\left[(\quad) - (\quad)\right]$$

$$= -\frac{2(s+2)}{s^2+4s+5} + \frac{2(s+1)}{s^2+2s+5}$$

$$\Rightarrow \mathcal{L}(t f(t)) = -\frac{2(s+2)}{(s+2)^2+1^2} + 2\frac{(s+1)}{(s+1)^2+2^2}$$

$$\Rightarrow \mathcal{L}(t f(t)) = \mathcal{L}\left(-2e^{-2t}\cos(t) + 2e^{-t}\cos(2t)\right)$$

$$\Rightarrow f(t) = \frac{2}{t}\left(e^{-t}\cos(2t) - e^{-2t}\cos(t)\right)$$

# 19 - 21 hints and 15

16 June 2021 14:57

bit.ly/ma-108 - Check the methods PDF, that has integrals similar to those in Q15.

Q.19 Show that if $f(t) = 1/(1+t^2)$ then its Laplace transform $F(s)$ satisfies the differential equation $F'' + F = 1/s$. Deduce that $F(s) = \left[\int_0^\infty \frac{\sin \lambda d\lambda}{(\lambda + s)}.\right]$

Q.20 Show that the Laplace transform of $\log t$ is $-s^{-1}\log s - Cs^{-1}$. Identify the constant $C$ in terms of the gamma function.

$C = -\Gamma'(1)$

Q.21 Evaluate the integral $\int_0^\infty \exp\left\{-\left(at + \frac{b}{t}\right)\right\}\frac{dt}{\sqrt{t}}$ where $a$ and $b$ are positive. Use this result to compute the Laplace transform of $\frac{1}{\sqrt{t}}\exp\left(\frac{-b}{t}\right)$., $b > 0$.

Show that (i) $\int_0^\infty (\quad)$ satisfies the ODE.

(ii) Thus, $F(s) = \int_0^\infty (\quad) + a\cos(s) + b\sin(s)$.

this has limit 0 as $s \to \infty$

$\therefore\ a = b = 0.$

(21) $I = \int_0^\infty \exp(-at - b/t)\, 1/\sqrt{t}\, dt.$

Let $\lambda = \sqrt{\frac{b}{a}}$. Then, $a\lambda = \frac{b}{\lambda} = \sqrt{ab}$.

Let $c = \sqrt{ab}$.

Put $t = \lambda u$ in $I$ to get.

$$I = \int_0^\infty \exp\left(-a\lambda u - \frac{b}{\lambda u}\right)\frac{\lambda}{\sqrt{\lambda}} \cdot \frac{1}{\sqrt{u}}\, du$$

$$= \sqrt{\lambda}\int_0^\infty \exp\left(-c\left(u + \frac{1}{u}\right)\right)\frac{1}{\sqrt{u}}\, du.$$

Put $u = 1/v$ and add to get.

$$2I = \sqrt{\lambda}\, e^{-2c}\int_0^\infty \exp\left(-c\left(\sqrt{u} - \frac{1}{\sqrt{u}}\right)^2\right)\left(1 + \frac{1}{u}\right)\frac{1}{\sqrt{u}}\, du.$$

Put $\sqrt{u} - 1/\sqrt{u} = w$ and solve.

Ans: $I = \sqrt{\lambda\pi/c}\; e^{-2c} = \sqrt{\pi/a}\; e^{-2\sqrt{ab}}.$

Q.15. Evaluate the following integrals by computing their Laplace transforms.

(i) $f(t) = \int_0^\infty \frac{\sin(tx)}{x}dx$ (ii) $f(t) = \int_0^\infty \frac{\cos tx}{x^2+a^2}dx$ (iii) $f(t) = \int_0^\infty \sin(tx^a)dx, \quad a > 1$

(iv) $\int_0^\infty \frac{1}{\ldots}(1-\cos tx)dx$ (v) $\int_0^\infty \frac{\sin^4 tx}{\ldots}dx$ (vi) $\int_0^\infty \left(\frac{x^2-b^2}{\ldots}\right)\frac{\sin tx}{\ldots}dx$

$$f(t) = \int_0^\infty \sin(tx^a)\,dx$$

$$F(s) = \int_0^\infty \frac{x^a}{s^2+x^{2a}}\,dx$$

$$x^{2a} = s^2 u$$

$$= \int_0^\infty \frac{s\sqrt{u}}{s^2(1+u)} \cdot \frac{s^{1/a}}{2a} \cdot u^{1/2a - 1}\,du$$

$$= \frac{1}{s^{1-1/a}} \cdot \frac{1}{2a} \int_0^\infty \frac{u^{1/2 + 1/2a - 1}}{1+u}\,du$$

$$F(s) = \underline{\underline{\frac{1}{s^{1-1/a}}}} \cdot \left[\frac{1}{2a}\,\pi \operatorname{cosec}\left(\frac{\pi}{2} + \frac{\pi}{2a}\right)\right]$$

$\downarrow$ take $\mathcal{L}^{-1}$

$$f(t) = \frac{\pi}{2a} \sec\left(\frac{\pi}{2a}\right) \frac{t^{-1/a}}{\Gamma(1-1/a)}$$

In particular, $\displaystyle\int_0^\infty \sin(x^2)\,dx = \frac{\sqrt{\pi}}{2\sqrt{2}}$

## Laplace of log:

$$f(t) = \log(t).$$

$$F(s) = \int_0^\infty \log(t)\, e^{-st}\,dt$$

$st = u$

$$= \int_0^\infty \log(u/s)\, e^{-u}\, \frac{du}{s}$$

$$= s^{-1} \underbrace{\int_0^\infty \log(u)\, e^{-u}\, du}_{-c} - \frac{\log(s)}{s} \underbrace{\int_0^\infty e^{-u} du}_{=1}$$

$$C = -\int_0^\infty \log(u)\, e^{-u}\, du.$$

Recall $\Gamma(x+1) = \int_0^\infty e^{-u}\, u^x\, du$

$$\frac{d}{dx}\Gamma(x+1) = \int_0^\infty e^{-u}\, \ln(u)\, u^x\, du$$

$$\Rightarrow \Gamma'(1) = \int_0^\infty e^{-v} \ln(v)\, dv$$

$$= -c$$

$$\therefore C = -\Gamma'(1)$$