ODEs TSC

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IIT Bombay

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The ODE is said to be linear if it of the form

$$a_n(x)y^{(n)}(x) + \cdots + a_0(x)y = b(x)$$

for some $n \geqslant 0$ and functions $a_0, \ldots, a_n, b$ of x.

Solutions

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Orthogonal trajectories

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Example: $x^2 + y^2 = \lambda^2$. Differentiating gives $x + yy' = 0$. Replacing y with $-1/y'$ gives

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Solving it gives $y = cx$ ($c \in \mathbb{R}$) as the family of orthogonal trajectories.

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$$H_1(x) + H_2(y) = c$$

for $c \in \mathbb{R}$.

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To solve: put $y = xv$ and things "magically" fall in place by becoming a separable ODE in v.

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Just integrate the above to get $k(y)$ and in turn, get $u(x, y)$.

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$$\mu = \exp\left(\int \frac{M_y - N_x}{N} dx\right).$$

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Let f be a function of one variable defined on some interval $I \subseteq \mathbb{R}$. f is said to be Lipschitz continuous if there exists some $L \geqslant 0$ such that

$$|f(x_1) - f(x_2)| \leqslant L|x_1 - x_2|$$

for all $x_1, x_2 \in I$.

Now, if f is a function of two variables defined on some $D \subseteq \mathbb{R}^2$, then we say that f is Lipschitz continuous with respect to y if there exists some $L \geqslant 0$ such that

$$|f(x, y_1) - f(x, y_2)| \leqslant L|y_1 - y_2|$$

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Remarks and examples

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An non-example of Lipschitz function (in y) is: $f(x, y) = \sqrt{|y|}$ defined on $[-1, 1] \times [-1, 1]$. Similarly, $f(x, y) = y^2$ is not Lipschitz w.r.t. y on $\mathbb{R}^2$ but is so on bounded domains.

Back to uniqueness

Theorem 5 (Uniqueness)

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As before, there may a solution on a larger interval. Moreover, there may still be a larger interval where the solution is unique.

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Under the assumptions of the existence-uniqueness theorem, the above converges to the solution y of the IVP defined by $y(x) := \lim_{n\to\infty} y_n(x)$.

Table of Contents

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Disclaimer

Our results will always assume that the ODE is in standard form. This is crucial.

Homogeneous

The standard ODE is said to be homogeneous if $b(x) = 0$, i.e., it is of the form

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From now on, "homogeneous" will refer to the above, not the one we had defined earlier.

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(Bernoulli) If the ODE was instead $y' + P(x)y = Q(x)y^n$ for some $n \neq 0, 1$, then substitute $v = y^{1-n}$ and it will "magically" get reduced to the above.

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Theorem 7 (Dimension result)

The solution space of (2) is a two-dimensional real vector space.

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Note that the Wronskian is defined for any two functions, without any mention of any DE.

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Again, note that no reference to any DE has been made.

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Consequently, if $x_0 \in I$, then

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A consequence of the earlier is the following: If y_1 is one solution of

$$y'' + p(x)y' + q(x)y = 0,$$

then a linearly independent solution y_2 to the above (*homogeneous*) equation is given by

$$y_2(x) = y_1(x) \int \frac{\exp\left(-\int p(x)\,dx\right)}{y_1(x)^2}\,dx.$$

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Note that basis being $\{y_1, y_2\}$ means that the general solution is given by $c_1 y_1 + c_2 y_2$ for $c_1, c_2 \in \mathbb{R}$.

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(Dimension result) The solution space of (3) is n-dimensional.

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The Wronskian of n nice function $y_1, \ldots, y_{n-1}$ is defined by

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Abel's formula

Theorem 12 (Abel-Liouville)

Let $y_1, \ldots, y_n$ be solutions of $y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_0(x)y = 0$.

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Note that the coefficient of $y^{(n-1)}$ is the one that appears above.

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We already know how to find the general solution of the homogeneous part. We now try to find a particular solution y_p of the non-homogeneous ODE.

Let μ be the multiplicity of m as a root of the characteristic polynomial. ($\mu = 0$ if m is not a root.) Then, the guess solution is

$$y_p = x^\mu (a_0 + a_1 x + \cdots + a_k x^k) e^{mx}.$$

The coefficients $a_0, \ldots, a_k$ are obtained by plugging y_p in (4) and comparing coefficients.

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The method of undetermined coefficients for Cauchy-Euler is the same with obvious modifications.

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Suppose we wish to solve

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For $c \geqslant 0$, define the Heaviside function u_c by

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Laplace of common functions

$f(t)$	$F(s)$	$f(t)$	$F(s)$
t	$1/s^2$	t^a	$\frac{\Gamma(a+1)}{s^{a+1}}$
$u_c(t)$	e^{-cs}/s	e^{at}	$\frac{1}{s-a}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$t\sin(\omega t)$	$\frac{2\omega s}{(s^2+\omega^2)^2}$	$t\cos(\omega t)$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$
$e^{at}\sin(\omega t)$	$\frac{\omega}{(s-a)^2+\omega^2}$	$e^{at}\cos(\omega t)$	$\frac{s-a}{(s-a)^2+\omega^2}$
$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$	$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$
$e^{at}\sinh(\omega t)$	$\frac{\omega}{(s-a)^2-\omega^2}$	$e^{at}\cosh(\omega t)$	$\frac{s-a}{(s-a)^2-\omega^2}$

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We have another theorems which says that if $F = \mathcal{L}(f)$, then $\lim_{s\to\infty} F(s) = 0$. For example, this rules out 1, $\sin(s)$, $\log(s^2+1)$, $\log(s^{-1})$ from being Laplace transforms.

Examples of some Laplace inverses

For $a \in \mathbb{R}$ and $n \geqslant 1$, we have

$$\mathcal{L}^{-1}\left(\frac{1}{(s-a)^n}\right) = \frac{1}{n!} e^{at} t^{n-1}.$$

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$$\mathcal{L}^{-1}\left(\frac{c_1(s-a)+c_2}{(s-a)^2+b^2}\right) = e^{at}\left(c_1\cos(bt) + \frac{c_2}{b}\sin(bt)\right).$$

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