ODEs TSC

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ODEs

We know what an ODE is. The order of an ODE is the order of the highest derivative in the equation.

$\sin\left(\frac{d^2y}{dx^2}\right) = \left(\frac{dy}{dx}\right)^3$ has order ____.

The ODE is said to be linear if it of the form

$$a_n(x)y^{(n)}(x) + \cdots + a_0(x)y = b(x)$$

for some $n \geqslant 0$ and functions $a_0, \ldots, a_n, b$ of x.

Solutions

Consider the ODE to be given as

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}).$$

For example, $y' = -x/y$.

An explicit solution of the above ODE on an interval I is a function ϕ defined on I such that

$$\phi^{(n)}(x) = f(x, \phi(x), \dots, \phi^{(n-1)}(x))$$

for all $x \in I$. Example: $\phi(x) = \sqrt{25 - x^2}$ on the interval $(-5, 5)$.

An implicit solution is a relation $g(x, y) = 0$ if this relation defines at least one function ϕ which is an explicit solution on some nonempty interval. Example: $x^2 + y^2 = 25$.

Orthogonal trajectories

Suppose we are given a family of curves, indexed by a parameter λ: $F(x, y, \lambda) = 0$. We wish to find the family of orthogonal trajectories.

First, differentiate the above and eliminate the parameter λ. This will now give you an equation involving x, y, y'. Replace y' with $-1/y'$. Solving this ODE now gives you the family of orthogonal trajectories.

Example: $x^2 + y^2 = \lambda^2$. Differentiating gives $x + yy' = 0$. Replacing y with $-1/y'$ gives

$$xy' = y.$$

Solving it gives $y = cx$ ($c \in \mathbb{R}$) as the family of orthogonal trajectories.

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Separable ODEs

An ODE of the form

$$M(x) + N(y)y' = 0$$

is called a separable ODE. It may also be suggestively written as

$$M(x)dx + N(y)dy = 0.$$

The above is solved by "simply integrating". More precisely, if H_1 and H_2 are functions such that $H_1'(x) = M(x)$ and $H_2'(y) = N(y)$, then the general solution is

$$H_1(x) + H_2(y) = c$$

for $c \in \mathbb{R}$.

Homogeneous functions

Recall that a function f of n-variables is called homogeneous of degree d if

$$f(tx_1, \ldots, tx_n) = t^d f(x_1, \ldots, x_n)$$

for all $t \neq 0$. Examples: $f(x, y) = (x - y)^2 + xy$,
$f(x, y) = y^2 + x^2 \exp(x/y)$.

Definition 1

The first order ODE

$$M(x, y) + N(x, y)y' = 0$$

is called homogeneous if M and N are homogeneous of equal degree.

To solve: put $y = xv$ and things "magically" fall in place by becoming a separable ODE in v.

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Definition

Definition 2

A first order ODE

$$M(x, y) + N(x, y)y' = 0$$

is called exact if there exists a function $u(x, y)$ such that

$$u_x = M \quad \text{and} \quad u_y = N.$$

The general solution to the above ODE is then $u(x, y) = c$ for $c \in \mathbb{R}$.

A necessary condition for the ODE to be exact is $M_y = N_x$.

The above is *also* sufficient if the domain is "nice": for example, if the domain is convex. (More generally, it suffices for the domain to be simply-connected, if you still remember what that means.)

Solving

The question is: how to find u? This is simple, just go by instincts.

You know that $u_x(x, y) = M(x, y)$. So, integrate M with respect to x. Remember that the arbitrary constant you add will be a function of y now. This will leave you with something like

$$u(x, y) = \int M(x, y)dx + k(y).$$

Now, differentiate the above with respect to y and equate it to $N(x, y)$. Things will "magically" get cancelled and you will be left with

$$k'(y) = \text{some function of } y.$$

Just integrate the above to get $k(y)$ and in turn, get $u(x, y)$.

Integrating Factors

Sometimes, the ODE $M(x, y)dx + N(x, y)dy = 0$ may not be exact. To combat this, we try to find an integrating factor, $\mu(x, y)$, such that the equation

$$\mu M dx + \mu N dy = 0$$

is exact. The above gives us the equation

$$\mu_y M + \mu M_y = \mu_x N + \mu N_x.$$

Now, we typically assume either $\mu_y = 0$ (or $\mu_x = 0$) and hope that the remaining terms cancel out nicely in a way that we are actually left with μ_x/μ being only a function of x (or the other way around). More precisely, if $\frac{M_y - N_x}{N}$ is a function of x, then we have an integrating factor μ given by

$$\mu = \exp\left(\int \frac{M_y - N_x}{N} dx\right).$$

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Definition and existence

Definition 3

An initial value problem (IVP) is an ODE of the form

$$y' = f(x, y),\ y(x_0) = y_0. \tag{1}$$

We now see a condition telling us when the above has a solution.

Theorem 4 (Existence)

Let R be a rectangle of the form $(x_0 - a, x_0 + a) \times (y_0 - b, y_0 + b)$.
Suppose that f is continuous and bounded on R, say $|f(x, y)| \leqslant K$ for all $(x, y) \in R$.
Then, (1) has an explicit solution defined on $(x_0 - \delta, x_0 + \delta)$, where $\delta := \min\{a, b/K\}$.

Note that a solution *may* exist on a larger interval. Furthermore, there may be multiple solutions on that given interval itself. We now see when the solution is unique.

Lipschitz

Let f be a function of one variable defined on some interval $I \subseteq \mathbb{R}$. f is said to be Lipschitz continuous if there exists some $L \geqslant 0$ such that

$$|f(x_1) - f(x_2)| \leqslant L|x_1 - x_2|$$

for all $x_1, x_2 \in I$.

Now, if f is a function of two variables defined on some $D \subseteq \mathbb{R}^2$, then we say that f is Lipschitz continuous with respect to y if there exists some $L \geqslant 0$ such that

$$|f(x, y_1) - f(x, y_2)| \leqslant L|y_1 - y_2|$$

for all $(x, y_1), (x, y_2) \in D$.

Remarks and examples

Any Lipschitz continuous function (of one variable) is continuous. Consequently, if f is Lipschitz continuous with respect to y, then for every fixed x, the function $f(x, y)$ is a continuous in y. However, f may not be continuous in x. For example,

$$f(x, y) = \lfloor x \rfloor + y$$

is Lipschitz continuous in y but $f(x, 1)$ is not continuous function.

If f is a differentiable function of one variable with f' bounded, then f is Lipschitz. Consequently, if f is a function of two variables with $\frac{\partial f}{\partial y}$ bounded, then f is Lipschitz with respect to y.

An non-example of Lipschitz function (in y) is: $f(x, y) = \sqrt{|y|}$ defined on $[-1, 1] \times [-1, 1]$. Similarly, $f(x, y) = y^2$ is not Lipschitz w.r.t. y on $\mathbb{R}^2$ but is so on bounded domains.

Back to uniqueness

Theorem 5 (Uniqueness)

Suppose that we have the IVP

$$y' = f(x, y),\ y(x_0) = y_0.$$

As before, suppose f is continuous on $R = (x_0 - a, x_0 + a) \times (y_0 - b, y_0 + b)$ and bounded by K. We already saw that the above IVP has *a* solution defined on $(x_0 - \delta, x_0 + \delta)$. Furthermore, if f also satisfies the Lipschitz condition with respect to y on R, then the solution is *unique* on that interval.

As before, there may a solution on a larger interval. Moreover, there may still be a larger interval where the solution is unique.

Picard's iteration method

As before, suppose we have the IVP: $y' = f(x, y)$, $y(x_0) = y_0$.

The above differential equation is equivalent to solving the integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt.$$

We define the Picard's iterates recursively as

$$\begin{aligned} y_0(x) &:= y_0, \\ y_{n+1}(x) &:= y_0 + \int_{x_0}^{x} f(t, y_n(t))\, dt. \end{aligned}$$

Under the assumptions of the existence-uniqueness theorem, the above converges to the solution y of the IVP defined by $y(x) := \lim_{n \to \infty} y_n(x)$.

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Definition and convention

We had seen what a linear ODE was. A linear ODE of degree n in standard form is one of the form

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y = b(x).$$

For example, $xy' - 10y = 0$ is *not* in standard form. However, if we are interested in solving the ODE on $(0, \infty)$, then we can put it in standard form as $y' - \frac{10}{x}y = 0$.

Disclaimer

Our results will always assume that the ODE is in standard form. This is crucial.

Homogeneous

The standard ODE is said to be homogeneous if $b(x) = 0$, i.e., it is of the form

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y = 0.$$

From now on, "homogeneous" will refer to the above, not the one we had defined earlier.

First order

A first order linear ODE is particular simple, it is of the form

$$y' + P(x)y = Q(x).$$

The above can be solved by multiplying with the integrating factor

$$\mu(x) := \exp\left(\int_{x_0}^{x} P(t)\,dt\right).$$

The final solution is also explicitly given by

$$y(x) = \frac{1}{\mu(x)}\left(\int Q(x)\mu(x)\,dx + c\right).$$

(Bernoulli) If the ODE was instead $y' + P(x)y = Q(x)y^n$ for some $n \neq 0, 1$, then substitute $v = y^{1-n}$ and it will "magically" get reduced to the above.

Second order

Consider the following second order homogeneous linear ODE:

$$y'' + p(x)y' + q(x)y = 0, \qquad (2)$$

where the functions p and q are continuous on some open interval I.

Theorem 6 (Existence-uniqueness result)

Let $x_0 \in I$, and fix $a, b \in \mathbb{R}$. There is a unique solution y, defined on I, satisfying (2) along with $y(x_0) = a$ and $y'(x_0) = b$.

Theorem 7 (Dimension result)

The solution space of (2) is a two-dimensional real vector space.

Wronskian

Definition 8

Let y_1 and y_2 be differentiable on I. The Wroskian of y_1 and y_2 is defined by

$$W(y_1, y_2)(x) := \det \begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix}.$$

Note that the Wronskian is defined for any two functions, without any mention of any DE.

Wronskian and linear dependence

Recall that two functions y_1 and y_2 are said to be linearly dependent (LD) on I if there exists constants $c_1, c_2 \in \mathbb{R}$ *not both zero* such that

$$c_1 y_1(x) + c_2 y_2(x) = 0$$

for all $x \in I$.

Theorem 9

If y_1 and y_2 are LD on I, then $W(y_1, y_2)(x) = 0$ for all $x \in I$.

However, even if $W(y_1, y_2)(x) = 0$ for all $x \in I$, it is **not** necessary that y_1 and y_2 are linearly dependent on I.

Consider $I = (-1, 1)$ and the functions $y_1(x) = x^3$ and $y_2(x) = |x|^3$.

Again, note that no reference to any DE has been made.

Wronskian, linear dependence, and an ODE

Now we make reference to an ODE and also see a (strong!) converse to the previous theorem.

Theorem 10

Let y_1 and y_2 be solutions to $y'' + p(x)y' + q(x)y = 0$ on an open interval I (as before, p and q are continuous on I). The following are equivalent:

1. y_1 and y_2 are linearly dependent on I.
2. Their Wronskian vanishes everywhere on I.
3. Their Wronskian vanishes at one point in I.

What the above theorem tells us about x^3 and $|x|^3$ is that they cannot be the solutions to a standard linear ODE on $(-1, 1)$. Note that they *are* solutions to $x^2y'' - 5xy' + 6y = 0$.

Similarly, x^2 and x^3 are not LD on $(-1, 1)$ but their Wronskian vanishes at 0. (Again, both of them are solutions to that non-standard ODE written above.)

Abel's formula

On the previous slide, we saw that if the Wronskian is nonzero at a point, then it must nonzero everywhere. We actually have a more precise relation given by Abel's formula. The notations I, p, q continue to be as before.

Theorem 11 (Abel-Liouville)

Let y_1 and y_2 be any two solutions of $y'' + p(x)y' + q(x)y = 0$. Then, the Wronskian $W := W(y_1, y_2)$ satisfies the differential equation

$$W'(x) = -p(x)W(x).$$

Consequently, if $x_0 \in I$, then

$$W(x) = W(x_0)\exp\left(-\int_{x_0}^{x} p(t)\,dt\right).$$

Getting a second solution

A consequence of the earlier is the following: If y_1 is one solution of

$$y'' + p(x)y' + q(x)y = 0,$$

then a linearly independent solution y_2 to the above (*homogeneous*) equation is given by

$$y_2(x) = y_1(x) \int \frac{\exp\left(-\int p(x)\, dx\right)}{y_1(x)^2}\, dx.$$

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Constant coefficients

ODE in question:

$$y'' + py' + qy = 0.$$

Here p and q are real numbers.

Solution: Find the roots of the quadratic $m^2 + pm + q = 0$. Call them m_1 and m_2.

Case 1: Roots are real and distinct. A basis for solution is $\{e^{m_1 x}, e^{m_2 x}\}$.
Case 2: Real repeated root. A basis for solution is $\{e^{m_1 x}, xe^{m_1 x}\}$.
Case 3: Roots are distinct and not real. In this case, the roots are of the form $a \pm \iota b$. A basis for solution is $\{e^{ax}\cos(bx), e^{ax}\sin(bx)\}$.

Note that basis being $\{y_1, y_2\}$ means that the general solution is given by $c_1 y_1 + c_2 y_2$ for $c_1, c_2 \in \mathbb{R}$.

Cauchy-Euler

ODE in question:

$$x^2y'' + pxy' + qy = 0.$$

Here p and q are real numbers. The above is **not** in standard form. However, we wish to solve the above on $(0, \infty)$, where it can be put in standard form by dividing by x^2.

Solution: Find the roots of the quadratic $m(m-1) + pm + q = 0$. Call them m_1 and m_2.

Case 1: Roots are real and distinct. A basis for solution is $\{x^{m_1}, x^{m_2}\}$.
Case 2: Real repeated root. A basis for solution is $\{x^{m_1}, x^{m_1}\log(x)\}$.
Case 3: Roots are distinct and not real. In this case, the roots are of the form $a \pm \iota b$. A basis for solution is $\{x^a\cos(b\log(x)), x^a\sin(b\log(x))\}$.

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We have the n-th order linear homogeneous ODE in standard form given by

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_0(x)y = 0. \tag{3}$$

Here, the coefficients $p_0, \ldots, p_{n-1}$ are assumed to be continuous on an open interval I.

(Existence-uniqueness) Let $x_0 \in I$. Suppose that $k_0, \ldots, k_{n-1}$ are arbitrary real numbers. (3) has a unique solution y, defined on I, such that $y(x_0) = k_0$, $y'(x_0) = k_1$, ..., $y^{(n-1)}(x_0) = k_{n-1}$.

(Dimension result) The solution space of (3) is n-dimensional.

Wronskian

The Wronskian of n nice function $y_1, \ldots, y_{n-1}$ is defined by

$$W(y_1, \ldots, y_n)(x) := \det \begin{bmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{bmatrix}.$$

Suppose $y_1, \ldots, y_n$ are solutions to the earlier homogeneous linear ODE in standard form, and $x_0 \in I$. Then, $y_1, \ldots, y_n$ are LD iff their Wronskian vanishes at x_0 iff their Wronskian vanishes everywhere on I.

Abel's formula

Theorem 12 (Abel-Liouville)

Let $y_1, \ldots, y_n$ be solutions of $y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_0(x)y = 0$. Then, the Wronskian $W := W(y_1, \ldots, y_n)$ satisfies the differential equation

$$W'(x) = -p_{n-1}(x)W(x).$$

Consequently, if $x_0 \in I$, then

$$W(x) = W(x_0)\exp\left(-\int_{x_0}^{x} p_{n-1}(t)\,dt\right).$$

Note that the coefficient of $y^{(n-1)}$ is the one that appears above.

Constant coefficients ODE

To solve:

$$y^{(n)} + p_{n-1}y^{(n-1)} + \cdots + p_0 y = 0,$$

where $p_0, \ldots, p_{n-1}$ are real numbers.

Method: Find the solutions of the characteristic equation

$$m^n + p_{n-1}m^{n-1} + \cdots + p_0 m = 0.$$

If m_0 is a root with multiplicity $k+1$ (here $k \geqslant 0$), then the solutions are $e^{m_0 x}, xe^{m_0 x}, \ldots, x^k e^{m_0 x}$. Since there are n roots with multiplicity (over $\mathbb{C}$), we get n LI solutions.

If $m_0 = a + \iota b$, then its conjugate is also a root. Replace $x^k e^{(a \pm \iota b)x}$ with $x^k e^{ax} \cos(bx)$ and $x^k e^{ax} \sin(bx)$.

Cauchy-Euler ODE

To solve:

$$x^n y^{(n)} + p_{n-1}x^{n-1}y^{(n-1)} + \cdots + p_0 y = 0,$$

where $p_0, \ldots, p_{n-1}$ are real numbers.

Method: Find the solutions of the characteristic equation

$$m(m-1)\cdots(m-(n-1)) + m(m-1)\cdots(m-(n-2))p_{n-1} + \cdots + p_0 m = 0.$$

If m_0 is a root with multiplicity $k+1$ (here $k \geqslant 0$), then the solutions are $x^{m_0}, x^{m_0}\log(x), \ldots, x^{m_0}(\log(x))^k$.

As before, in case of complex roots, we have the following replacement: $x^{a\pm\iota b}(\log(x))^k \rightsquigarrow x^a\cos(b\log(x))(\log(x))^k, x^a\sin(b\log(x))(\log(x))^k$.

Method of Undetermined Coefficients

Consider the non-homogeneous ODE

$$y^{(n)} + p_{n-1}y^{(n-1)} + \cdots + p_0 y = x^k e^{mx}, \qquad (4)$$

where $p_0, \ldots, p_{n-1}$ are real numbers.

We already know how to find the general solution of the homogeneous part. We now try to find a particular solution y_p of the non-homogeneous ODE.

Let μ be the multiplicity of m as a root of the characteristic polynomial. ($\mu = 0$ if m is not a root.) Then, the guess solution is

$$y_p = x^{\mu}(a_0 + a_1 x + \cdots + a_k x^k)e^{mx}.$$

The coefficients $a_0, \ldots, a_k$ are obtained by plugging y_p in (4) and comparing coefficients.

Method of Undetermined Coefficients

Instead of e^{mx}, we may have $e^{ax}\sin(bx)$ or $e^{ax}\cos(bx)$. In this case, the guess is of the form

$$y_p = x^{\mu}(a_0 + a_1x + \cdots + a_kx^k)e^{ax}\cos(bx) + x^{\mu}(b_0 + b_1x + \cdots + b_kx^k)e^{ax}\sin(bx).$$

Alternately, you may want to break the problem of $e^{ax}\sin(bx)$ into two complex problems of $e^{(a+\iota b)x}$ and $e^{(a-\iota b)x}$.

The method of undetermined coefficients for Cauchy-Euler is the same with obvious modifications.

Method of Variation of Parameters

Suppose we wish to solve

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_0(x)y = r(x),$$

and we already have LI solutions $y_1, \ldots, y_n$ of the homogeneous part.

Then, a particular solution is given by

$$y_p = v_1 y_1 + \cdots + v_n y_n,$$

where $v_1, \ldots, v_n$ are determined by solving

$$\begin{bmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{bmatrix} \begin{bmatrix} v_1'(x) \\ v_2'(x) \\ \vdots \\ v_n'(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ r(x) \end{bmatrix}.$$

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Definition

Definition 13

Let $f : (0, \infty) \to \mathbb{R}$ be a function. The Laplace transform of f, denoted $\mathcal{L}(f)$, is defined by

$$\mathcal{L}(f)(s) := \int_0^\infty e^{-st} f(t)\, dt.$$

This function is typically defined on (a, ∞) for some $a > 0$. The Laplace transform of a function of t is typically written as a function of s, using the corresponding capital letter.

If f is piecewise continuous and of exponential order, then $\mathcal{L}(f)(s)$ exists for s large enough.

More precisely: if there exist $a, t_0, K > 0$ such that $|f(t)| \leqslant Ke^{at}$ for all $t > t_0$, then $\mathcal{L}(f)(s)$ exists for all $s > a$.

Heaviside and Convolution

For $c \geqslant 0$, define the Heaviside function u_c by

$$u_c(t) := \begin{cases} 0 & t < c, \\ 1 & t \geqslant c. \end{cases}$$

The convolution of two functions f and g defined on $(0, \infty)$ is defined by

$$(f * g)(t) := \int_0^t f(\tau) g(t - \tau)\, d\tau.$$

Note that $f * g$ is itself a new function. $*$ is commutative, associative, and distributes over addition. $1 * f = f$ is **not** true in general.

Properties of Laplace

Linearity: $\mathcal{L}(af + bg) = a\mathcal{L}(f) + b\mathcal{L}(g)$ for functions f, g and reals a, b.

Shifting I: If $\mathcal{L}(f(t)) = F(s)$, then $\mathcal{L}(e^{at}f(t)) = F(s - a)$.

Shifting II: $\mathcal{L}(u_c(t)f(t - c)) = e^{-cs}F(s)$, where $c \geqslant 0$.

Scaling: $\mathcal{L}(f(ct)) = \frac{1}{c}F\left(\frac{s}{c}\right)$.

Derivative I: $\mathcal{L}(f')(s) = sF(s) - f(0)$, $\mathcal{L}(f'')(s) = s^2F(s) - sf(0) - f''(0)$.

Derivative II: $\mathcal{L}(tf(t)) = -F'(s)$.

Convolution: $\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g)$.

Laplace of common functions

$f(t)$	$F(s)$	$f(t)$	$F(s)$
t	$1/s^2$	t^a	$\frac{\Gamma(a+1)}{s^{a+1}}$
$u_c(t)$	e^{-cs}/s	e^{at}	$\frac{1}{s-a}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$t\sin(\omega t)$	$\frac{2\omega s}{(s^2+\omega^2)^2}$	$t\cos(\omega t)$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$
$e^{at}\sin(\omega t)$	$\frac{\omega}{(s-a)^2+\omega^2}$	$e^{at}\cos(\omega t)$	$\frac{s-a}{(s-a)^2+\omega^2}$
$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$	$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$
$e^{at}\sinh(\omega t)$	$\frac{\omega}{(s-a)^2-\omega^2}$	$e^{at}\cosh(\omega t)$	$\frac{s-a}{(s-a)^2-\omega^2}$

Inverse Laplace transforms

Lerch's theorem tells us that if f and g are good enough functions with $\mathcal{L}(f) = \mathcal{L}(g)$, then $f(t) = g(t)$ at all points of continuity of f and g.

It then makes sense to talk about $\mathcal{L}^{-1}$. It is checked that $\mathcal{L}^{-1}$ is also linear.

We have another theorems which says that if $F = \mathcal{L}(f)$, then $\lim_{s\to\infty} F(s) = 0$. For example, this rules out 1, $\sin(s)$, $\log(s^2 + 1)$, $\log(s^{-1})$ from being Laplace transforms.

Examples of some Laplace inverses

For $a \in \mathbb{R}$ and $n \geqslant 1$, we have

$$\mathcal{L}^{-1}\left(\frac{1}{(s-a)^n}\right) = \frac{1}{n!}e^{at}t^{n-1}.$$

Similarly,

$$\mathcal{L}^{-1}\left(\frac{c_1(s-a)+c_2}{(s-a)^2+b^2}\right) = e^{at}\left(c_1\cos(bt) + \frac{c_2}{b}\sin(bt)\right).$$

Sometimes, it may be useful to use derivatives. For example, if we wish to compute the Laplace inverse of $F(s) = \log\left(\frac{s^2+1}{s^2+4}\right)$, we note that $F'(s) = \frac{2s}{s^2+1} - \frac{2s}{s^2+4}$. Now, we can take Laplace inverse and using $\mathcal{L}(tf(t)) = -F'(s)$, we get the desired f.