

→ Hi. I'm Anyaman. Almost an alumnus.

→ [bit.ly/ma-108](https://bit.ly/ma-108) (Recordings of recap and tut - MS Team)

PDFs of whatever I write

MA-108

ODE: A relation involving  $x, y, y', \dots, y^{(n)}$ .

$$F(x, y, \dots, y^{(n)}) = 0.$$

$$y^{(n)} = G(x, y, \dots, y^{(n-1)}) \quad (*)$$

interested in

eg. ①  $y' + y = 0$   
 ②  $(y')^2 + x e^y + \sin(y'') = 0$

NOT

NOT example:  $y(y(x)) = y'(x)$ .

EXPLICIT: (explicit)

A solution of (\*) is a function  $\phi$  which is defined on some (open) interval  $I$  s.t.

$$\phi^{(n)}(x) = G(x, \phi(x), \dots, \phi^{(n-1)}(x)) \quad \text{for all } x \in I.$$

I.V.P:

$$\begin{cases} y' = f(x, y) \\ y(x_0) = y_0 \end{cases}$$

Here, we will try to find an interval  $I$  containing  $x_0$  s.t.  $\exists \phi : I \rightarrow \mathbb{R}$  satisfying the above.

Linear ODE: Is an ODE of the form:

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_0(x)y = b(x).$$

The above is said to have order  $n$  if  $a_n(x) \neq 0$ .

EXAMPLE:  $(1-x^2)y'' - 2xy' + n(n+1)y = 0 \quad (n \in \mathbb{Z})$

NON-EXAMPLES: ①  $yy' + x = 4^3$

EXAMPLE:  $(1-x)y' - x^2y = 0$

NON-EXAMPLES: ①  $yy' + x = 4^3$

②  $e^y = y'$

Legendre's equation  
(MA 207)

EXPLICIT SOLUTION:

IVP:  $y' = e^{x^2}; y(0) = 0.$

$\phi(x) := \int_0^x e^{t^2} dt$  is an EXPLICIT SOLUTION.

IMPLICIT:

"Don't have  $y = \phi(x)$ .  
Rather, a relation  $F(x, y) = 0$   
s.t. on some interval  $I$ , we can  
get a  $\phi$  s.t.  $F(x, \phi(x)) = 0$   
 $\forall x \in I$ ."

EXAMPLE: ①  $yy' + x = 0.$

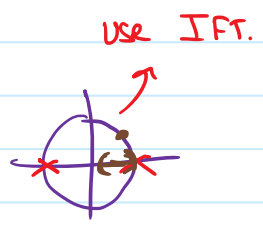
Particular  $y^2 + x^2 = c$

$y^2 + x^2 = 4 \rightarrow$  Implicit

But this

$y = \sqrt{4-x^2}$  and  $y = -\sqrt{4-x^2}$

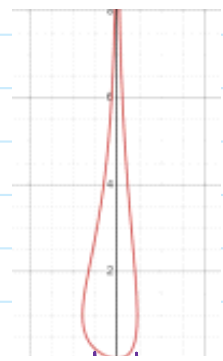
as two solutions.



②  $[F(x, y) = y \sin x + x^2 e^y - y]$

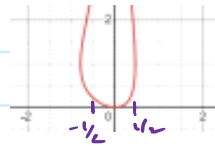
Let's say  $y \sin x + x^2 e^y - y = 0$

From the graph, it is "clear" that we can get a function of  $x$  on  $(-1/2, 1/2)$ .



Formally: IMPLICIT FUNCTION THEOREM

# Formally: IMPLICIT FUNCTION THEOREM



Note: ①  $F(0, 0) = 0$ .

$$\textcircled{2} \frac{\partial F}{\partial y}(0, 0) = (\sin(x) + x^2 e^y - 1) \Big|_{(x,y)=(0,0)} = -1 \neq 0.$$

The above conditions tell you that you can get  $y$  as a function of  $x$  locally.

$$F(x, y) = c$$

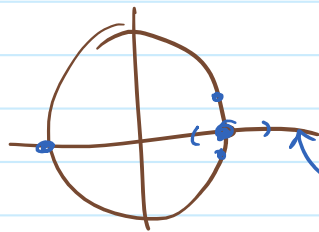
Given  $(x_0, y_0)$  s.t.  
 $F(x_0, y_0) = c$ .

$$x^2 + y^2 = 4$$

$\downarrow \frac{\partial}{\partial y}$   
 $2y$



Q: Can you get  $\phi$  on some interval  $I$  around  $x_0$  s.t.  
 $F(x, \phi(x)) = c$   
 $\forall x \in I$ .



A: (IFT) Yes, if  $\frac{\partial F}{\partial y}(x_0, y_0) \neq 0$ .

# Week 2

18 May 2022 12:41

1. Integrating factor
2. Homogeneous ODEs  $\rightsquigarrow$  substitute  $v = y/x$
3. Orthogonal trajectory  $\rightsquigarrow$  example
4. Lipschitz continuity  $\rightsquigarrow$

## 1. Integrating Factors.

$$M dx + N dy = 0. \quad (1)$$

$\hookrightarrow$  functions of  $x$  and  $y$

(1) is said to be **exact** if  $M_y = N_x$ .  
(I will assume domain is nice.  
For example, convex (or simply-connected, more generally).)

We can then solve the above, i.e., we get a function  $\phi$  s.t.  $\phi_x = M$  and  $\phi_y = N$ .  
Then,  $\phi = C$  is the gen. solution.

If (1) is NOT exact, we try to find an integrating factor  $\mu$  s.t.

$$\mu M dx + \mu N dy = 0$$

is exact.

Then, we want

$$(\mu M)_y = (\mu N)_x$$

or

$$\mu_y M + \mu M_y = \mu_x N + \mu N_x.$$

For simplicity, we assume  $\mu$  is a function of (say)  $x$  alone. Then  $\mu_y \equiv 0$  and we get

$$\mu_x N = \mu (M_y - N_x) \quad \text{or}$$

$$\frac{\mu_x}{\mu} = \frac{M_y - N_x}{N}.$$

4. Let  $D \subseteq \mathbb{R}^2$  (say a rectangle).  
 $f: D \rightarrow \mathbb{R}$ .

$f$  is said to be Lipschitz (in  $y$ ) if  
 $\exists M \geq 0$  s.t.

$$|f(x, y_1) - f(x, y_2)| \leq M |y_1 - y_2|$$

for all  $(x, y_1), (x, y_2) \in D$ .

Let  $I \subseteq \mathbb{R}$ .  $g: I \rightarrow \mathbb{R}$  is said to be Lipschitz if  $\exists L \geq 0$  s.t.

$$|g(x_1) - g(x_2)| \leq L |x_1 - x_2|$$

for all  $x_1, x_2 \in I$ .

Example: •  $I = \mathbb{R}$ ,  $g(x) = x$ . ( $L = 1$  works.)

•  $I = [1, 2]$ ,  $g(x) = x^2$ .

$$\frac{|g(x_1) - g(x_2)|}{|x_1 - x_2|} = |x_1 + x_2| \leq 4.$$

$L = 4$  works

•  $I = [0, 1]$ ,  $g(x) = \sqrt{x}$ .

Not Lipschitz.

Suppose not. Then,  $\exists L \geq 0$  s.t.

$$|g(x) - g(0)| \leq L |x - 0| \quad \forall x \in I.$$

$$\Leftrightarrow \sqrt{x} \leq Lx \quad \forall x \in I$$

$$\Rightarrow \frac{1}{\sqrt{x}} \leq L \quad \text{for all } x \in I \setminus \{0\}.$$

But this is a contradiction, choose

$$x = \frac{1}{L^2 + 1} \in I \setminus \{0\}.$$

Digression:

→ complete metric space.

$f: X \rightarrow X$  is conti

s.t.

$$|f(x) - f(y)| \leq \alpha |x - y|$$

for some  $0 < \alpha < 1$ .

Then,  $f$  has a unique fixed pt,

$$x \in X \text{ s.t. } f(x) = x.$$

$$\text{DE: } y' = f(x, y)$$

$$y \in \mathbb{R}$$

$$\phi(x) = \int_{x_0}^x f(s, \phi(s)) ds$$

$$\|f - g\| = \sup_{x \in [0,1]} |f(x) - g(x)|.$$

$X =$  appropriate metric space

$$f: X \rightarrow X$$

$$f(\phi) := \int_{x_0}^x f(s, \phi(s)) ds.$$

Show  $f$  is a contraction.

Done.

MA 108  
DI-T3

- Constant coefficient (second order) linear ODE
- Cauchy Euler (second order) ODE

①  $y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0$   
↳ looking to solve this on  $\mathbb{R}$ .

②  $x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \dots + a_1 x y' + a_0 y = 0$   
↳ looking to solve this on  $(0, \infty)$ .

In both cases:  $a_0, \dots, a_{n-1} \in \mathbb{R}$ .

To solve ①, we substitute  $y = e^{mx}$ .

②  $y = x^m$ .

General theory tells that ① and ② have an  $n$ -dimensional solution space.

Our "algorithm" actually gives us  $n$  distinct linearly independent solutions. Then, we have found all.

ALGORITHM: ① Put  $y = e^{mx}$ . We end up with an  $n$ -degree polynomial in ' $m$ ', given as:

$$m^n + a_{n-1} m^{n-1} + \dots + a_1 m + a_0 = 0.$$

In  $\mathbb{C}$ , the above has  $n$  solutions (counted with multiplicity).  
If  $m_0$  is a solution with multiplicity  $k+1$  ( $k \geq 0$ ), then we have the solutions

(P.D)  $(D - m_0)^{k+1} y = 0$   
 $(D - m_0)^{k+1} y = 0$

$e^{m_0 x}, x e^{m_0 x}, \dots, x^k e^{m_0 x}$   
↳  $k+1$  linearly indep.

If  $m_0$  is a non-real  $n$  complex, then  $\bar{m}_0$  is a root with same multiplicity

$e^{a x} \cos(b x)$   
 $e^{a x} \sin(b x)$

$\left\{ \begin{array}{l} e^{m_0 x}, \\ e^{\bar{m}_0 x} \end{array} \right\}$

$\left\{ \begin{array}{l} x e^{m_0 x}, \dots, x^k e^{m_0 x} \\ x e^{\bar{m}_0 x}, \dots, x^k e^{\bar{m}_0 x} \end{array} \right\}$

$m_0 = a + ib$   
 $a, b \in \mathbb{R}$

$x e^{a x} \cos(b x), x e^{a x} \sin(b x), \dots$

② Here the  $n$ -degree polynomial is:

$$m(m-1) \dots (m-(n-1)) + a_{n-1} m(m-1) \dots (m-(n-2)) + \dots + a_1 m + a_0 = 0.$$

Again:  $n$  solutions in  $\mathbb{C}$  ...

$m_0$  has mult.  $k+1$ , then the functions:

$$x^{m_0}, x^{m_0} \log x, \dots, x^{m_0} (\log x)^k.$$

If  $m_0 \in \mathbb{C} \setminus \mathbb{R}$ , then write  $m_0 = a + ib$

$$\begin{aligned} x^{m_0} &= \exp(m_0 \cdot \log x) \\ &= \exp(a \log x + i b \log x) \\ &= x^a \cdot [\cos(b \log x) + i \sin(b \log x)] \end{aligned}$$

A before  $\bar{m}_0$  is a root ...

$$\begin{array}{l} x^a \cos(b \log x), \quad x^a \cos(b \log x) \log x, \dots \\ x^a \sin(b \log x), \quad x^a \sin(b \log x) \log x, \dots \end{array}$$