Q.11.

18 May 2022 12:36

Q.11. ① Solve the differential equation $\sqrt{1-y^2}dx + \sqrt{1-x^2}dy = 0$ with the conditions $y(0) = \frac{\pm 1}{2}\sqrt{3}$. ② Sketch the graphs of the solutions and show that they are each arcs of the same ellipse. ③ Also show that after these arcs are removed, the remaining part of the ellipse does not satisfy the differential equation.

① Rearranging: $$\frac{1}{\sqrt{1-x^2}}\,dx + \frac{1}{\sqrt{1-y^2}}\,dy = 0.$$

Integrating gives $\sin^{-1}(x) + \sin^{-1}(y) = c.$

Corresponding to the different initial of $y(0) = \pm\frac{\sqrt{3}}{2}$, we get $c = \pm\frac{\pi}{3}$ (Corresponding to the same sign).

Solutions: $$\begin{cases} \sin^{-1}x + \sin^{-1}y = \pi/3, \\ \sin^{-1}x + \sin^{-1}y = -\pi/3. \end{cases}$$

take cos

② $$\cos(\sin^{-1}x + \sin^{-1}y) = \cos(\pm\pi/3) = 1/2$$

$$\Leftrightarrow \sqrt{1-x^2}\sqrt{1-y^2} - xy = 1/2$$

$$\Leftrightarrow \sqrt{1-x^2}\sqrt{1-y^2} = 1/2 + xy \quad \text{— } (*)$$

$$\Rightarrow (1-x^2)(1-y^2) = (1/2 + xy)^2$$

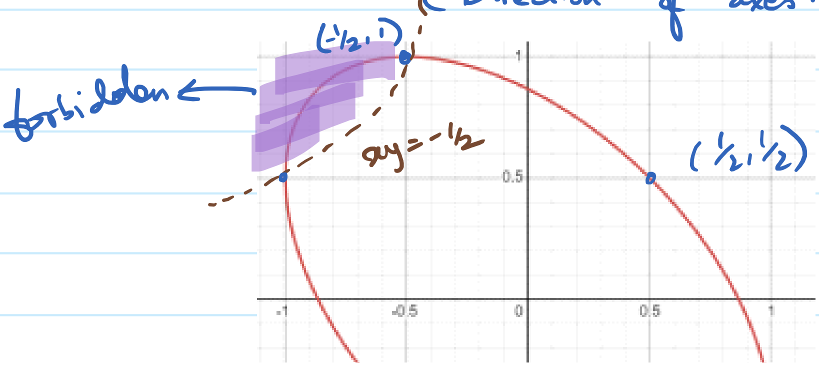
$$\Leftrightarrow x^2 + y^2 + xy = 3/4$$

$$\Leftrightarrow 3\left(\frac{x+y}{\sqrt{2}}\right)^2 + \left(\frac{x-y}{\sqrt{2}}\right)^2 = \frac{3}{2}.$$

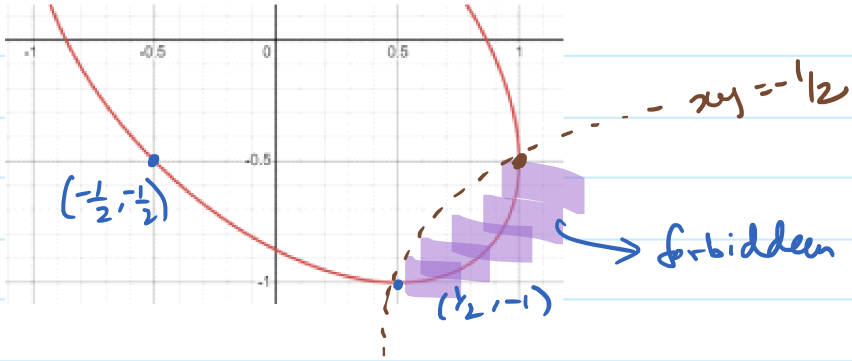
MA106 diagonalise $\begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}$

e-vectors

(Direction of axes: $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$ and $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$)



xy = -1/2



③ ① (*) shows that $\frac{1}{2} + xy$ has to be nonnegative for the equation to be a solution.

Thus, the highlighted purple parts are not solutions.

② From the ODE, we see $\frac{dy}{dx} = -\frac{\sqrt{\ }}{\sqrt{\ }} \leq 0.$

But the purple highlight have $\frac{dy}{dx} > 0.$

Q.2.

18 May 2022 12:38

Q.2. Solve the following exact equation

(iii) $e^x y(x+y)dx + e^x(x+2y-1)dy = 0$

We need ϕ s.t. $\phi_x = e^x y(x+y)$ — (1) and

$\phi_y = e^x(x+2y-1)$. — (2)

Integrating (1) wrt x: $\phi = y(x-1)e^x + e^x y^2 + f(y)$.

Now, diff wrt y: $\phi_y = (x-1)e^x + 2ye^x + f'(y)$.

Compare with (2):

$$e^x(x+2y-1) = (x-1)e^x + 2ye^x + f'(y)$$

$$\Rightarrow f'(y) = 0.$$

Thus, the gen. solution is

$$y(x-1)e^x + e^x y^2 = c.$$

Q.3.

18 May 2022 12:39

Q.3. Determine (by inspection suitable) Integrating Factors (IF's) so that the following equations are exact.

(v) $(2x + e^y)dx + xe^y dy = 0$, (vi) $(x^2 + y^2)dx + xydy = 0$

$\partial/\partial y \to e^y$ $\quad \partial/\partial x \to e^y$

(v) Already exact.

(vi) Let us derive the IF μ.

$$\left(\mu (x^2+y^2)\right)_y = (\mu xy)_x$$

$$\Rightarrow \mu_y \cdot (x^2+y^2) + 2\mu y = \mu_x \cdot xy + \mu \cdot y$$

$$\Rightarrow \mu_y (x^2+y^2) + \mu y = \mu_x \cdot xy.$$

(Putting $\mu_x = 0$ does not seem to help.
But $\mu_y = 0$ does.)

Assume μ is a function of x alone.

Then, $\mu y = \mu_x \cdot xy$

$$\Rightarrow \frac{\mu_x}{\mu} = \frac{1}{x}.$$

Solve to get $\mu = x$ as IF.

Q.7.

18 May 2022 12:41

Q.7. Solve the following homogeneous equation.

(iv) $xy' = y + x\cos^2\frac{y}{x}$

We have $\dfrac{dy}{dx} = \dfrac{y}{x} + \cos^2(y/x)$. ——(1)

Put $y = vx$.

Then, $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$.

Put in (1) to get

$$v + x\frac{dv}{dx} = v + \cos^2(v)$$

$$\Rightarrow \sec^2(v)\frac{dv}{dx} = \frac{1}{x}$$

$$\Rightarrow \tan(v) = \log|x| + C$$

SUBSTITUTE $v = y/x$ back

$$\boxed{\tan(y/x) = \log|x| + C.}$$

Q.8.

18 May 2022 12:43

Q.8. Solve the following first order linear equation.

(iv) $y' = \operatorname{cosec} x + y \cot x$.

$y' + P(x)y = Q(x)$

$IF = e^{\int P(x)\,dx}$

$$y' - (\cot x)\, y = \operatorname{cosec}(x).$$

$$IF : \exp\left(\int -\cot(x)\,dx\right)$$

$$= \exp\left(-\log|\sin x|\right)$$

$$= \operatorname{cosec}(x)$$

Multiplying with above gives

$$\underbrace{\left(\operatorname{cosec}(x)\, y' - \operatorname{cosec}(x)\cot(x)\, y\right)}_{=\,(\operatorname{cosec}(x)\, y)'} = \operatorname{cosec}^2(x)$$

$$\Rightarrow \operatorname{cosec}(x)\, y = \int \operatorname{cosec}^2(x)\,dx$$

$$= -\cot(x) + C$$

$$\Rightarrow \boxed{y = -\cos(x) + C\sin(x)}$$

Q.9.

18 May 2022 12:44

$y' + f(x)y = g(x)$ ✓ $\alpha = 0$

$y' + (f(x) - g(x))\, y = 0$ ✓ $\alpha = 1$

separable as well!

Q.9. A differential equation of the form $y' + f(x)y = g(x)y^\alpha$ is called a Bernoulli equation. Note that if $\alpha = 0$ or 1 it is linear and for other values it is nonlinear. Show that the transformation $y^{1-\alpha} = u$ converts it into a linear equation. Use this to solve the following equation.

$$\text{(iv)}\,(xy + x^3y^3)\frac{dy}{dx} = 1.$$

slides

recap

$$y' + f(x)\, y = g(x)\, y^\alpha.$$

$$y^{1-\alpha} = u$$

$$(1-\alpha)\, y^{-\alpha} y' = u',$$

$$\Rightarrow y' = y^\alpha \cdot u' \cdot \frac{1}{(1-\alpha)}$$

divide by y^α

$$\frac{u'}{1-\alpha} + f(x)\, u = g(x)$$

$$u' + (1-\alpha)\, f(x)\, u = (1-\alpha)\, g(x)$$

(iv) $$(xy + x^3y^3)\frac{dy}{dx} = 1 \qquad \left[\, y' + f(x)y = g(x)y^\alpha \right]$$

$$\Rightarrow x\, y\frac{dy}{dx} + x^3y^3\frac{dy}{dx} = 1$$

The above is NOT of the given form.
However, changing our perspective will help!

Rearranging gives

$$\frac{dx}{dy} = xy + x^3y^3$$

$$\Rightarrow \frac{dx}{dy} + (-y)\, x = y^3 \cdot x^3.$$

Bernoulli with 'x' and 'y' interchanged.

Bernoulli with 'x' and 'y' interchanged.

$\alpha = 3$.

Substitution: $x^{-2} = u$

$$\Rightarrow (-2)\, x^{-3} \frac{dx}{dy} = \frac{du}{dy}$$

$$\Rightarrow \frac{1}{(-2)\,x^{-3}} \cdot \frac{du}{dy} + (-y)\, x = y^3\, x^3$$

$$\Rightarrow \frac{1}{-2} \cdot \frac{du}{dy} + (-y)\, x^{-2} = y^3$$

$$\Rightarrow -\frac{1}{2} \frac{du}{dy} - yu = y^3$$

$$\Rightarrow \frac{du}{dy} + (2y)\, u = -2y^3.$$

$$IF = e^{\int 2y\, dy} = e^{y^2}$$

$$\Rightarrow e^{y^2} u = \int -2y^3 e^{y^2}\, dy$$

$$= - \int y^2 e^{y^2} (2y)\, dy \qquad y^2 = t$$

$$= - \int t\, e^t\, dt$$

$$= -(t-1)\, e^t + c$$

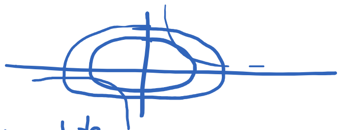
$$= -(y^2-1)\, e^{y^2} + c$$

$$\Rightarrow u = -(y^2-1) + c e^{-y^2}$$

$$\Rightarrow \boxed{x^{-2} = 1 - y^2 + c e^{-y^2}}.$$

Q.12.

18 May 2022 12:45



hyperbola ellipse empty set

Q.12. ① Find the ODE for the family of curves $\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$, $(0<b<a)$ and ② find the ODE for the orthogonal trajectories. ③ Solve this ODE for orthogonal trajectories.

- Parameter in family is only λ. (a and b are fixed.)

① Need to eliminate λ.

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1 \qquad \text{—— (1)}$$

$\partial/\partial x$

$$\frac{x}{a^2+\lambda}+\frac{yy'}{b^2+\lambda}=0$$

$$\Rightarrow \frac{b^2+\lambda}{a^2+\lambda}=-\frac{yy'}{x}$$

$$\Rightarrow 1+\frac{b^2-a^2}{a^2+\lambda}=\frac{-yy'}{x}$$

$$\Rightarrow \left(1+\frac{yy'}{x}\right)=\frac{a^2-b^2}{(a^2+\lambda)}$$

$$\Rightarrow a^2+\lambda=\frac{a^2-b^2}{1+\frac{yy'}{x}}=\frac{x(a^2-b^2)}{x+yy'}$$

$$\Rightarrow a^2+\lambda=\frac{x(a^2-b^2)}{x+yy'} \qquad \text{—— (2)}$$

(add b^2-a^2

$$\Rightarrow b^2+\lambda=\frac{(b^2-a^2)yy'}{x+yy'} \qquad \text{—— (3)}$$

Plug (2) and (3) into (1) to get:

$$\frac{x^2(x+yy')}{(a^2-b^2)\ x} - \frac{y^2(x+yy')}{(a^2-b^2)\ yy'} = 1$$

$$\Rightarrow \quad x(x+yy') - \frac{y(x+yy')}{y'} = a^2-b^2$$

$$\Rightarrow \quad \boxed{\frac{(xy'-y)(x+yy')}{y'} = a^2-b^2.}$$

— (ODE 1)

② To get ODE for orthogonal trajectory, replace y' in (ODE 1) with $-1/y'$.

This gives: $$\frac{(-x/y' - y)(x - y/y')}{-1/y'} = a^2-b^2$$

$$\Rightarrow \quad \frac{(-x-yy')(xy'-y)}{-1\cdot y'} = a^2-b^2$$

$$\Rightarrow \quad \boxed{\frac{(x+yy')(xy'-y)}{y'} = a^2-b^2}$$

— (ODE 2)

NOTE: (ODE1) $\equiv$ (ODE2)!

③ The same family is the solution.

(Paradox?)