Q.1.

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Q.1. Classify the following equations (order, linear or non-linear):

(i) $\frac{d^3y}{dx^3} + 4(\frac{dy}{dx})^2 = y$ (ii) $\frac{dy}{dx} + 2y = \sin x$ (iii) $y\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + y = 0$

(iv) $1\frac{d^4y}{dx^4} + (\sin x)\frac{dy}{dx} + x^2y = 0$. (v) $(1+y^2)\frac{d^2y}{dt^2} + t\frac{d^6y}{dt^6} + y = e^t$.

functions of x

	ORDER	LINEAR	
(i)	3	×	$(dy/dx)^2$
(ii)	1	✓	$1 \cdot y' + 2 \cdot y = \sin(x)$
(iii)	2	×	$y \cdot y''$
(iv)	4	✓	
(v)	6	×	
$y'' \cdot y'' + y''' = 0$	3	×	

Q.2.

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Q.2. Formulate the differential equations represented by the following functions by eliminating the arbitrary constants a, b and c:

(v) $y = a\sin x + b\cos x + a$ (vii) $y = cx + f(c)$.

Also state the order of the equations obtained.

(vii) $y = cx + \underline{\underline{f(c)}}$ $\quad \frac{d}{dx}$

$\Rightarrow \quad y' = c$

<u>Final:</u> $\quad y = y'x + f(y')$.

(v) $y = a\sin x + b\cos x + a \quad —(0) \quad \frac{d}{dx}$

$\Rightarrow \quad y' = a\cos x - b\sin x \quad —(1) \quad d/dx$

$\Rightarrow \quad y'' = -a\sin x - b\cos x \quad —(2)$

(1) and (2) can be written as

$$\begin{bmatrix} \cos x & -\sin x \\ -\sin x & -\cos x \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y' \\ y'' \end{bmatrix}$$

det $= -1$

$$\Rightarrow \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \cos x & \sin x \\ \sin x & -\cos x \end{bmatrix} \begin{bmatrix} y' \\ y'' \end{bmatrix}$$

$$\Rightarrow \begin{cases} a = (\cos x)\, y' + (\sin x)\, y'' \\ b = (\sin x)\, y' - (\cos x)\, y''. \end{cases}$$

Plug in (0).

Q.3.

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Q.3. Solve the equation $x^3(\sin y)y' = 2$. Find the particular solution such that $y(x) \to \frac{\pi}{2}$ as $x \to +\infty$.

↳ SEPARABLE

Separating gives: $(\sin y)\, y' = \dfrac{2}{x^3}$ integrate

$$\Rightarrow -\cos y = -\frac{1}{x^2} - C$$

$$\Rightarrow \boxed{\cos y = \frac{1}{x^2} + C}$$ General solution.

↳ NOT explicit

$$y(x) = \cos^{-1}\left(\frac{1}{x^2} + C\right)$$ is EXPLICIT.

assuming C is "appropriate"

Let us find the desired particular solution.

$$\lim_{x \to \infty} y(x) = \lim_{x \to \infty} \cos^{-1}\left(\frac{1}{x^2} + C\right)$$

$\cos^{-1}(\cdot)$ is continuous

$$= \cos^{-1}(0 + C)$$
$$= \cos^{-1}(C).$$

For the above to be $\pi/2$, we have $C = 0$.

Thus, the desired solution is

$$\boxed{y(x) = \cos^{-1}\left(\frac{1}{x^2}\right).}$$

Q.5.

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e^{mx} is an e-vector of D. $D(f) := \frac{df}{dx}$

x^m is for xD $\quad x\,D(x^m) = m x^m$.

Q.5. Find the values of m for which

LINEAR ODES

(a) $y = e^{mx}$ is a solution of
(i) $y'' + y' - 6y = 0$ → CONSTANT COEFFICIENTS | SIMPLY PLUG IT IN.

(b) $y = x^m$ for $x > 0$ is a solution of
(ii) $x^2y''' - xy'' + y' = 0$. → "Euler's" equation

$$a_n x^n y^{(n)} + \cdots + a_1 x y' + a_0 y = 0. \quad a_0, \ldots, a_n \in \mathbb{R}$$

(a) (i) $(e^{mx})'' + (e^{mx})' - 6(e^{mx}) = 0$ → order 2

$$\Rightarrow m^2 e^{mx} + m e^{mx} - 6e^{mx} = 0$$ cancel e^{mx}

$$\Rightarrow m^2 + m - 6 = 0$$

$$\Rightarrow (m-2)(m+3) = 0$$

Thus, $m \in \{2, -3\}$. → DESIRED SET $\quad e^{2x}$, e^{-3x}

(b) (ii) $x^2 y''' - xy'' + y' = 0$ → order 3

$y = x^m$

$$x^2 (x^m)''' - x(x^m)'' + (x^m)' = 0$$

$$\Rightarrow x^2 \left[m(m-1)(m-2)\, x^{m-3}\right] - x\left[m(m-1)\, x^{m-2}\right] + m x^{m-1} = 0$$

$$\Rightarrow m(m-1)(m-2)\, x^{m-1} - m(m-1)\, x^{m-1} + m x^{m-1} = 0$$ cancel x^{m-1} since $x>0$

$$\Rightarrow m(m-1)(m-2) - m(m-1) + m = 0$$

$$\Rightarrow m\left[m^2 - 3m + 2 - m + 1 + 1\right] = 0$$

$$\Rightarrow m\left[m^2 - 4m + 4\right] = 0$$

$$\Rightarrow m(m-2)^2 = 0$$

Thus, $m \in \{0, 2\}$. only two solutions (distinct) x^0, x^2

(Since the order is 3, we would desire one more (lin. indep.) solution.)

[SPOILER: Check that $x^2 \log(x)$ is ALSO a solution.]

Q.7.

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Q.7. Let φ_i be a solution of $y' + ay = b_i(x)$ for $i = 1, 2$.

Show that $\varphi_1 + \varphi_2$ satisfies $y' + ay = b_1(x) + b_2(x)$. Use this result to find the solutions of $y' + y = \sin x + 3\cos 2x$ passing through the origin.

$$y' + ay = b_1(x) \rightsquigarrow \varphi_1(x)$$
$$y' + ay = b_2(x) \rightsquigarrow \varphi_2(x)$$

Given:
$$\varphi_1'(x) + a\varphi_1(x) = b_1(x) \quad — (1)$$
$$\varphi_2'(x) + a\varphi_2(x) = b_2(x) \quad — (2)$$

To show: $(\varphi_1 + \varphi_2)'(x) + a(\varphi_1 + \varphi_2)(x) = b_1(x) + b_2(x)$.

Add (1) and (2) to get the desired equation.

[Note: $(\varphi_1 + \varphi_2)'(x) = \varphi_1'(x) + \varphi_2'(x)$
$a(\varphi_1 + \varphi_2)(x) = a\varphi_1(x) + a\varphi_2(x)$]

We set up two equations:

① $y' + y = \sin(x)$
② $y' + y = 3\cos(2x)$

Solution of ①: Use the IF e^x to get

$$e^x(y' + y) = e^x \sin(x)$$
$$\Rightarrow \frac{d}{dx}(y e^x) = e^x \sin(x)$$
$$\Rightarrow y e^x = \frac{e^x}{2}(\sin x - \cos x) + a$$
$$\Rightarrow \boxed{y_1 = \frac{1}{2}(\sin x - \cos x) + ae^{-x}.}$$

$$\Rightarrow \boxed{y_1 = \frac{1}{2}(\sin x - \cos x) + ae^{-x}.}$$

Solution of ②: Use same IF.

We get

$$\boxed{y_2 = \frac{3}{5}(\cos 2x + 2\sin 2x) + be^{-x}.}$$

Finally, we get

$$y = y_1 + y_2$$

$$= \frac{1}{2}(\sin x - \cos x) + \frac{3}{5}(\cos 2x + 2\sin 2x) + (a+b)e^{-x}.$$

We want $y(0) = 0$.

Thus,

$$y(0) = 0 \Rightarrow -\frac{1}{2} + \frac{3}{5} + (a+b) = 0$$

$$\Rightarrow a+b = \frac{1}{2} - \frac{3}{5} = -\frac{1}{10}.$$

Thus, the solution passing through origin is

$$\boxed{y = \frac{1}{2}(\sin x - \cos x) + \frac{3}{5}(\cos 2x + 2\sin 2x) - \frac{e^{-x}}{10}.}$$

Q.8.

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$y(\pi/2) = 1.$
y is going to be continuous.
By continuity, $y(x) > 0$ on an interval around $\pi/2$.

Q.8. Obtain the solution of the following differential equations:

(ii) $y' = y \cot x;\ \ y(\pi/2) = 1$

(iv) $(x+2)y' - xy = 0;\ \ y(0) = 1$

(ii) $y' = y\cot(x).$

$\downarrow$ separable

(All logs are base e.)

$$\frac{y'}{y} = \cot(x)$$

$$\Rightarrow \log|y| = \log|\sin x| + C$$

$$\Rightarrow |y| = e^C |\sin x|$$

$\rightsquigarrow$ NOT EXPLICIT

Let us use the initial datum: (Put $x = \pi/2$.)

$$1 = e^C \sin(\pi/2) = e^C.$$

Thus, $C = 0.$

As an explicit solution, we get

$$y(x) = \sin(x) \quad \text{for} \quad x \in (0, \pi).$$

The absolute value sign is dropped using initial datum.

(iv) $(x+2)y' - xy = 0$; $y(0) = 1.$

$\big\}$ separable

$$\frac{y'}{y} = \frac{x}{x+2} = \frac{x+2-2}{x+2}$$

$$\frac{y'}{y} = \frac{x}{x+2} = \frac{x+2-2}{x+2}$$

$$= 1 - \frac{2}{x+2}.$$

Integrate: $\log|y| = x - 2\log|x+2| + C.$

$y(0) > 0$ so, $y(x) > 0$ around 0

$x+2 > 0$ around 0

$$\log(y) = x - 2\log(x+2) + C$$

$$\Rightarrow y = e^x \cdot \frac{1}{(x+2)^2} \cdot e^c.$$

Put $x=0$

$$\Rightarrow 1 = 1 \cdot \frac{1}{2^2} \cdot e^c$$

$$\Rightarrow e^c = 4.$$

Thus, $y(x) = \dfrac{4e^x}{(x+2)^2}$ is the desired solution.