Linear Algebra TSC

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IIT Bombay

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1. Matrices
2. EROs, ERMs, RREFs, and more
3. Vector spaces
4. Ranks of a matrix
5. Inner products
6. Eigenvectors and eigenvalues
7. Normal matrices and Spectral Theorems

Matrices

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Corresponding to each of the operations above, there are obvious row operations that can be performed on A^+, called the elementary row operations (EROs).

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It is fairly straightforward to perform EROs to turn A into an REF (and further an RREF).

Application of Gauss to solving linear equations

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Note: The above algorithm does not require prior knowledge that A is invertible. If you perform it on a non-invertible matrix, you'll end up finding out that it is not invertible.

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Suppose that $\mathbf{w}_1, \ldots, \mathbf{w}_k \in \mathbb{K}^n$ are arbitrary elements. Then, the set of all linear combinations

$$V := \{a_1\mathbf{w}_1 + \cdots a_k\mathbf{w}_k : a_1, \ldots, a_k \in \mathbb{K}\}$$

is a vector subspace.

Linear span

We can create vector subspaces out of linear combinations.

Suppose that $\mathbf{w}_1, \ldots, \mathbf{w}_k \in \mathbb{K}^n$ are arbitrary elements. Then, the set of all linear combinations

$$V := \{a_1\mathbf{w}_1 + \cdots a_k\mathbf{w}_k : a_1, \ldots, a_k \in \mathbb{K}\}$$

is a vector subspace.

We write $V = \text{span}\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ and say that V is spanned (or generated) by $\mathbf{w}_1, \ldots, \mathbf{w}_k$.

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Rephrasing slightly, linear independence means that

$$a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k = \mathbf{0} \Rightarrow a_1 = \cdots = a_k = 0.$$

Basis

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The size of B is called the dimension of V, denoted $\dim(V)$.

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3. S spans V.

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The above was the *First Fundamental Lemma in linear algebra*.

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1. Matrices
2. EROs, ERMs, RREFs, and more
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5. Inner products
6. Eigenvectors and eigenvalues
7. Normal matrices and Spectral Theorems

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Row rank = Column rank = number of pivots in any REF.

The common quantity above is called the rank of A.

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On the other hand, if you are asked to find a basis for the row space, then you pick the nonzero rows from any REF of A.

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What if "infinitely many" is replaced with "unique"?

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$$A = \begin{bmatrix} \boxed{1} & 3 & 1 & 1 \\ 0 & 0 & \boxed{2} & 4 \end{bmatrix}.$$

Note that we wish to solve $A\mathbf{x} = \mathbf{0}$ to get the null space. Thus, the equations obtained are

$$2x_3 + 4x_4 = 0 \quad \text{and} \quad x_1 + 3x_2 + x_3 + x_4 = 0.$$

The free variables are x_2 and x_4.

First solution: $x_2 = 1$ and $x_4 = 0$: We get $x_3 = 0$ and $x_1 = -3$.

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Thus, we get a basis as $\left\{\begin{bmatrix}-3 & 1 & 0 & 0\end{bmatrix}^{\mathsf{T}}, \begin{bmatrix}1 & 0 & -2 & 1\end{bmatrix}^{\mathsf{T}}\right\}$.

An example (continued)

Suppose that we were asked to find the general solution set of

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The general solution now is $\mathbf{x}_0 + \mathcal{N}(A)$. We had already found a basis earlier. Thus, the complete <u>set</u> of solutions is given by

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} : x_2, x_4 \in \mathbb{R} \right\}.$$

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True/False: Let A be a square matrix such that $A\mathbf{x} = \mathbf{0}$ has a unique solution, and fix $\mathbf{b} \in \mathbb{K}^n$. Does $A\mathbf{x} = \mathbf{b}$ also have a unique solution?

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Applications of determinants

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$$\mathbf{x} = \frac{(\operatorname{adj}(A))\mathbf{b}}{\det(A)}.$$

The above is essentially Cramer's rule.

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2. EROs, ERMs, RREFs, and more
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4. Ranks of a matrix
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Similarly, one has Pythagoras theorem and Cauchy Schwarz: $|\langle \mathbf{v}_1, \mathbf{v}_2 \rangle| \leqslant \|\mathbf{v}_1\| \cdot \|\mathbf{v}_2\|$.

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for all $1 \leqslant k \leqslant n$.

The idea is to do the following:

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Any orthogonal set of nonzero vectors is linearly independent.
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The polynomial $p(x) = \det(A - xI)$ is called the characteristic polynomial of A. The above proposition says that the eigenvalues of A are precisely the roots of the characteristic polynomial.

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Let λ be an eigenvalue of an $n \times n$ matrix A. Let g and m denote the geometric and algebraic multiplicities of λ respectively. Then,

$$1 \leqslant g \leqslant m \leqslant n.$$

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If A satisfies either (and hence, both) condition, then A is said to be diagonalisable over $\mathbb{K}$.

Criteria for diagonalisability

Let $A \in \mathbb{K}^{n \times n}$ be a square matrix, and let $\lambda_1, \ldots, \lambda_k \in \mathbb{K}$ be the distinct eigenvalues of A.

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Example: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is diagonalisable over $\mathbb{C}$ but not over $\mathbb{R}$.

The P and D

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Also note that a diagonal matrix is Hermitian iff it is real.
Similarly, a diagonal matrix is skew-Hermitian iff it is purely imaginary.

Spectral theorem

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The converse of the above is true as well: if A is unitarily diagonalisable (or has an orthonormal eigenbasis), then A is normal.

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Theorem 25

Let $A \in \mathbb{R}^{n\times n}$ be symmetric. Then, there exists an orthogonal matrix $O \in \mathbb{R}^{n\times n}$ such that $O^\top AO$ is diagonal.