MA 106 Endsem*

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Spring 2022

If nothing is mentioned, assume that similarity and eigenvalues/eigenvectors are being considered over $\mathbb{C}$. The characteristic polynomial of a square matrix A is defined as $p_A(t) = \det(A - tI)$.

1. Let A be a 2×2 real matrix with $\det(A) < 0$. Then,
   (a) A is diagonalisable over $\mathbb{R}$.
   (b) A is not diagonalisable over $\mathbb{R}$.
   (c) The given information is not sufficient to conclude.
2. Let A and B be 2×2 matrices with same eigenvalue(s) with the same geometric and algebraic multiplicities.
   True/False: A and B are similar.
3. Let A be a nonzero square matrix such that $A^k = O$ for some $k \geqslant 2$. Show that A is not diagonalisable.
4. Let A be a 9×7 matrix and B be a 4×3 matrix.
   Show that there exists a nonzero 7×4 matrix X such that $AXB = O$.
5. Let A and B be $n \times n$ matrices. Consider the following statements.
   (S1) A is similar to B.
   (S2) A and B have the same characteristic polynomial.
   (S3) $\det(A) = \det(B)$.

   Pick the correct options.
   (a) (S1) $\Rightarrow$ (S2)
   (b) (S2) $\Rightarrow$ (S3)
   (c) (S3) $\Rightarrow$ (S1)
   (d) (S1) $\Leftarrow$ (S2)
   (e) (S2) $\Leftarrow$ (S3)
   (f) (S3) $\Leftarrow$ (S1)

*Of course, this is not the actual endsem paper.

6. Let A and B be square matrices with the same characteristic polynomial. Suppose that for each eigenvalue, the geometric and algebraic multiplicities are the same for A and B.
   True/False: A and B are similar.

7. Let A be a 3×3 matrix with eigenvectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ corresponding to eigenvalues 0, 1, 2 respectively.
   Show that $A\mathbf{x} = \mathbf{u}$ has no solution.

8. Let A be a solved Sudoku interpreted as a 9×9 real matrix. Let $p(t)$ be the characteristic polynomial of A. Show that $p(45) = 0$.

9. Let A be an $n \times n$ matrix with characteristic polynomial $(-1)^n(t-1)(t-2)\cdots(t-n)$. Show that

$$A\mathbf{x} = \begin{bmatrix} 1 \\ 4 \\ \vdots \\ n^2 \end{bmatrix}$$

   has a solution.

10. Let A and B be 3×3 polynomials with characteristic polynomial $-t^3 + 6t^2 - 11t + 6$. Are A and B necessarily similar?

11. Let A and B be 3×3 matrices with characteristic polynomial $-t(t-1)^2$. Are A and B necessarily similar?

12. Let A be an $m \times n$ real matrix. Show that $\mathcal{N}(A^{\mathsf{T}}A) = \mathcal{N}(A)$.

13. Given $A = \begin{bmatrix} 6.5 & -2.5 & 2.5 \\ -2.5 & 6.5 & -2.5 \\ 0 & 0 & 4 \end{bmatrix}$, find a matrix B such that $B^2 = A$.

14. Let $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$. Prove that

$$\det \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix} = \prod_{1 \leqslant i < j \leqslant n} (\lambda_j - \lambda_i).$$

15. Let A be an $n \times n$ matrix satisfying $A^2 = A$. Suppose that A is neither the zero matrix nor the identity matrix.
    Choose the correct option(s).
    (a) A must be invertible.
    (b) A cannot be invertible.
    (c) The only possible eigenvalues of A are 0 and 1.
    (d) The null space and column space of A have a nonzero vector in common.

16. Show that if A is an $n \times n$ matrix satisfying $A^2 = A$, then A is diagonalisable. Conclude that if $A^2 = cA$ for some $c \neq 0$, then too A is diagonalisable.

17. Let A be a matrix such that $A^k = O$ for some $k \geqslant 1$. Show that $I - A$ is invertible.

18. Let A and B be 4×4 matrices defined by

$$A = \begin{bmatrix} 2 & 3 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 4 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Mark the correct option(s).

(a) Both A and B have the same characteristic polynomial.

(b) Both A and B have the same eigenvalues and their geometric multiplicities are also the same.

(c) Both A and B have the same eigenvalues and their algebraic multiplicities are also the same.

(d) A and B are similar.

19. Consider

$$A = \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 3 & 0 & 0 & 3 \\ 0 & -1 & 0 & 2 \end{bmatrix}, \mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \text{ and } \mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 1 \end{bmatrix}.$$

Choose the correct option(s):

(a) $A\mathbf{x} = \mathbf{u}$ has a solution.

(b) $\mathbf{v}$ is in the column space of A.

(c) None of the above.

20. Find the value(s) of k for which the system

$$\begin{aligned} y + 3kz &= 0 \\ x + 2y + 6z &= 2 \\ kx + 2ky + 12z &= -4 \end{aligned}$$

has no solution.

21. Let A be an $m \times n$ matrix. Let $\mathcal{N}(A)$, $\mathcal{R}(A)$, and $\mathcal{C}(A)$ denote the null space, row space, and column space of A, respectively. Pick the correct option(s).

(a) $\dim(\mathcal{N}(A)) = \dim(\mathcal{R}(A))$.

(b) $\dim(\mathcal{N}(A)) + \dim(\mathcal{R}(A)) = n$.

(c) $\dim(\mathcal{N}(A)) + \dim(\mathcal{C}(A)) = n$.

(d) $\mathcal{N}(A)$ and $\mathcal{R}(A)$ are orthogonal.

(e) $\mathcal{N}(A)$ and $\mathcal{C}(A)$ are orthogonal.

Recall that subspaces $V, W \subseteq \mathbb{R}^n$ are said to be orthogonal if $\langle v, w \rangle = 0$ for all $v \in V$ and all $w \in W$.

22. Let A be a self-adjoint matrix. Show that if $\langle A\mathbf{x}, \mathbf{x} \rangle = 0$ for all $\mathbf{x} \in \mathbb{C}^n$, then $A = O$.

23. Show that if $\|A\mathbf{x}\| = \|A^*\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{C}^n$, then A is a normal matrix.

24. Show that if $\|A\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{C}^n$, then A is a unitary matrix.

25. Which of the following matrices are diagonalisable?

$$\begin{bmatrix} 5 & -1 \\ 1 & 3 \end{bmatrix}, \quad \begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

26. Find necessary and sufficient conditions on a, b, c for the following matrix to be diagonalisable:

$$\begin{bmatrix} 2 & a & b \\ 0 & 1 & c \\ 0 & 0 & 2 \end{bmatrix}.$$

27. Let $\lambda \in \mathbb{C}$. Show that λ is an eigenvalue of A iff $\overline{\lambda}$ is an eigenvalue of A^*.

28. Use Gram-Schmidt to orthonormalise the ordered subset

$$\left(\begin{bmatrix} 1 & -1 & 2 & 0 \end{bmatrix}^{\mathsf{T}}, \begin{bmatrix} 1 & 1 & 2 & 0 \end{bmatrix}^{\mathsf{T}}, \begin{bmatrix} 3 & 0 & 0 & 1 \end{bmatrix}\right)^{\mathsf{T}}$$

and obtain an ordered orthonormal set $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$. Also, find $\mathbf{v}_4$ such that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is an orthonormal basis for $\mathbb{R}^4$.
Express $\begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}^{\mathsf{T}}$ as a linear combination of these four basis vectors.

29. Write down the symmetric matrix A such that the quadric

$$7x^2 + 7y^2 - 2z^2 + 20yz - 20zx - 2xy = 36$$

can be expressed as

$$\begin{bmatrix} x & y & z \end{bmatrix} A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 36.$$

Find an orthogonal matrix O such that $O^{\mathsf{T}}AO$ is diagonal.

30. Let A be an $n \times n$ normal matrix and $\lambda \in \mathbb{C}$.
Show that $A - \lambda I$ is a normal matrix.
Show that if $A\mathbf{x} = \lambda\mathbf{x}$, then $A^*\mathbf{x} = \overline{\lambda}\mathbf{x}$.

31. Give an example of a square matrix over $\mathbb{C}$ that is not diagonalisable.

32. Suppose $A \in \mathbb{R}^{3\times 3}$ satisfies $A^3 - 2A^2 = A - 2I$ and has the property that $\det(A) < 0$ and $\mathrm{trace}(A) > 2$.
Find the characteristic polynomial $p(t) = \det(A - tI)$.

It is **not** $-(t^3 - 2t^2 - t + 2)$.