TSC - QP

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QP

$\{1, x, x^2, \dots\}$ $\xrightarrow{GS}$ Legendre $\qquad \int_{-1}^{1} PQ$

$u_1, u_2, u_3 \longrightarrow$ orthonormal

Put them in a matrix as rows

$$A = \begin{bmatrix} - u_1 - \\ - u_2 - \\ - u_3 - \end{bmatrix} \rightarrow \text{find } \mathcal{N}(A)$$

$$Y^\perp = \{ v \in V : \langle v, y \rangle = 0 \text{ for all } y \in Y \}.$$

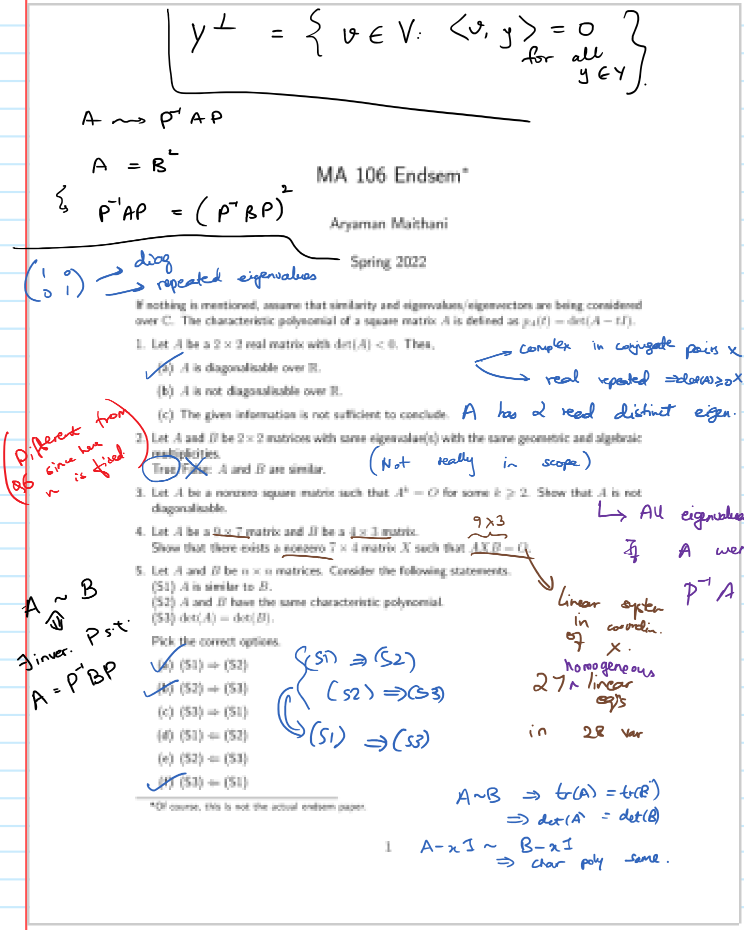
$A \rightsquigarrow P^{-1}AP$

$A = B^2$

$P^{-1}AP = (P^{-1}BP)^2$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \to$ diag

$\to$ repeated eigenvalues



MA 106 Endsem*

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If nothing is mentioned, assume that similarity and eigenvalues/eigenvectors are being considered over $\mathbb{C}$. The characteristic polynomial of a square matrix A is defined as $p_A(t) = \det(A - tI)$.

1. Let A be a 2×2 real matrix with $\det(A) < 0$. Then,
   - [✓ (a)] A is diagonalisable over $\mathbb{R}$.
   - (b) A is not diagonalisable over $\mathbb{R}$.
   - (c) The given information is not sufficient to conclude.

   $\to$ complex in conjugate pairs ✗

   $\to$ real repeated $\Rightarrow \det(A) \geq 0$ ✗

   A has 2 real distinct eigen.

2. Let A and B be 2×2 matrices with same eigenvalue(s) with the same geometric and algebraic multiplicities.
   [✓ True]/False: A and B are similar. (Not really in scope)

   (Different from Q6 since here n is fixed.)

3. Let A be a nonzero square matrix such that $A^k = O$ for some $k \geq 2$. Show that A is not diagonalisable.

   $\hookrightarrow$ All eigenvalues of A are 0.

   If A were diag., then $P^{-1}AP = O$.

   But then $A = POP^{-1} = O$. Contradiction.

4. Let A be a 9×7 matrix and B be a 4×3 matrix.
   Show that there exists a nonzero 7×4 matrix X such that $\underbrace{AXB}_{9 \times 3} = O$.

   $\to$ Linear system in coordin. of X.

   27 homogeneous linear eq's in 28 var

5. Let A and B be $n \times n$ matrices. Consider the following statements.
   (S1) A is similar to B.
   (S2) A and B have the same characteristic polynomial.
   (S3) $\det(A) = \det(B)$.

   Pick the correct options.
   - [✓ (a)] (S1) $\Rightarrow$ (S2)
   - [✓ (b)] (S2) $\Rightarrow$ (S3)
   - (c) (S3) $\Rightarrow$ (S1)
   - (d) (S1) $\Leftarrow$ (S2)
   - (e) (S2) $\Leftarrow$ (S3)
   - [✓ (f)] (S3) $\Leftarrow$ (S1)

   (S1) $\Rightarrow$ (S2)

   (S2) $\Rightarrow$ (S3)

   (S1) $\Rightarrow$ (S3)

$A \sim B$
$\Updownarrow$
$\exists$ inver. P s.t.
$A = P^{-1}BP$

$A \sim B \Rightarrow tr(A) = tr(B)$

$\Rightarrow \det(A) = \det(B)$

$A - xI \sim B - xI$

$\Rightarrow$ char poly same.

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*Of course, this is not the actual endsem paper.

1

Suppose λ is an eigen. of A.

$Av = \lambda v$ for some $v \neq 0$.

$\vdots$

$A^k v = \lambda^k v$

$A^k = O$

$\Rightarrow \lambda^k v = 0$ but $v \neq 0$.

Thus, $\lambda^k = 0$.

Thus, $\lambda = 0$.

$A_{m \times n}$
$B_{n \times p}$
$\text{rank}(AB) = \text{rank}(B^T A^T) \leq \text{rank}(B^T) = \text{rank}(B)$

Column space $(A) = \{Ax : x \in \mathbb{R}^n\}$

$\cup\!|$

Column space $(AB) = \{A(Bv) : v \in \mathbb{R}^p\}$

$\Rightarrow \dim(\mathcal{C}(AB)) \leq \dim(\mathcal{C}(A))$, where $\dim(\mathcal{C}(AB)) = \text{rank}(AB)$ and $\dim(\mathcal{C}(A)) = \text{rank}(A)$

wrt alg. mult.

Sum of eigen (in $\mathbb{C}$) = trace

Product of eigen (in $\mathbb{C}$) = dete.

$$A = \begin{bmatrix} \lambda & 1 \\ & \lambda \end{bmatrix} \longrightarrow (x - \lambda)^2$$

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \not\sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

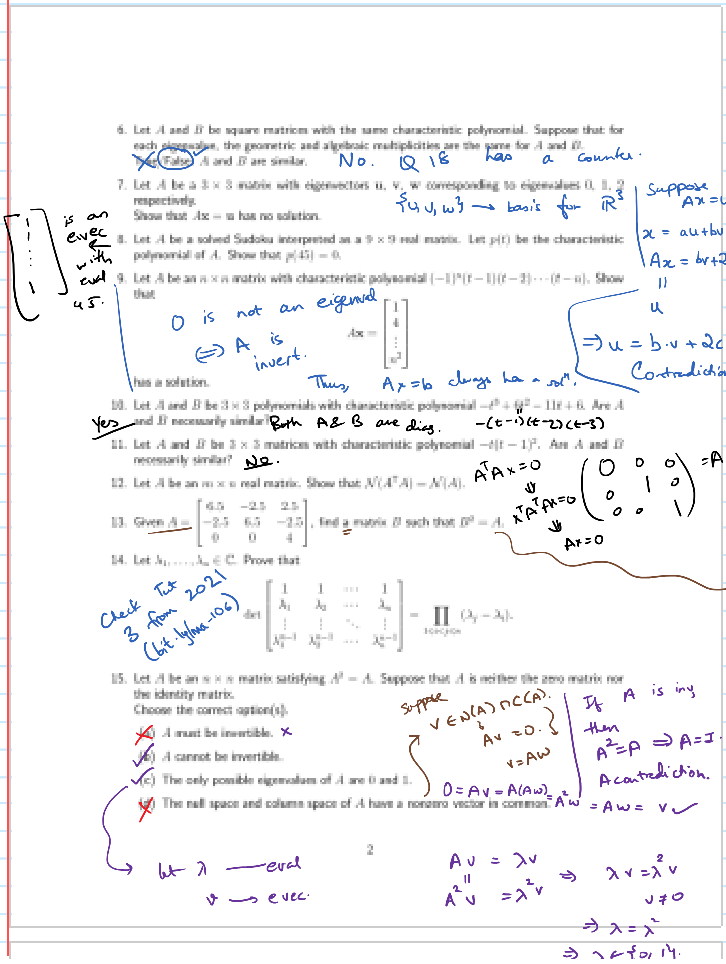
Same char poly.

Same det

$\text{rank}(A - \lambda I) = \text{rank} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 1$

$g_\lambda = \text{nullity}(A - \lambda I) = 1.$

$m_\lambda = 2.$ $\qquad A$ is NOT diagonalisable.



6. Let A and B be square matrices with the same characteristic polynomial. Suppose that for each eigenvalue, the geometric and algebraic multiplicities are the same for A and B.
   True/[✓ False]: A and B are similar. No. Q 18 has a counter.

7. Let A be a 3×3 matrix with eigenvectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ corresponding to eigenvalues 0, 1, 2 respectively.
   Show that $A\mathbf{x} = \mathbf{u}$ has no solution.

   $\{u, v, w\} \to$ basis for $\mathbb{R}^3$.

   Suppose $Ax = u$.

   $x = au + bv + cw.$

   $Ax = bv + 2cw = u$

   $\Rightarrow u = b \cdot v + 2c \cdot w$

   Contradiction.

8. Let A be a solved Sudoku interpreted as a 9×9 real matrix. Let $p(t)$ be the characteristic polynomial of A. Show that $p(45) = 0$.

   $\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ is an evec with eval 45.

9. Let A be an $n \times n$ matrix with characteristic polynomial $(-1)^n(t-1)(t-2)\cdots(t-n)$. Show that
   $$A\mathbf{x} = \begin{bmatrix} 1 \\ 4 \\ \vdots \\ n^2 \end{bmatrix}$$
   has a solution.

   0 is not an eigenval $\Leftrightarrow$ A is invert.

   Thus, $Ax = b$ always has a sol^n.

10. Let A and B be 3×3 polynomials with characteristic polynomial $-t^3 + 6t^2 - 11t + 6$. Are A and B necessarily similar? Yes. Both A & B are diag. $= -(t-1)(t-2)(t-3)$

11. Let A and B be 3×3 matrices with characteristic polynomial $-t(t-1)^2$. Are A and B necessarily similar? No.

    $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = A \qquad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

12. Let A be an $m \times n$ real matrix. Show that $\mathcal{N}(A^T A) = \mathcal{N}(A)$.

    $A^T A x = 0 \Downarrow x^T A^T A x = 0 \Downarrow Ax = 0$

13. Given $A = \begin{bmatrix} 6.5 & -2.5 & 2.5 \\ -2.5 & 6.5 & -2.5 \\ 0 & 0 & 4 \end{bmatrix}$, find a matrix B such that $B^2 = A$.

    Check that A is diagon...

    Find P s.t.

    $P^{-1}AP = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

    Take $B = P \begin{bmatrix} \sqrt{a} & & \\ & \sqrt{b} & \\ & & \sqrt{c} \end{bmatrix} P^{-1}$.

    $B^2 = P \begin{bmatrix} a & & \\ & b & \\ & & c \end{bmatrix} P^{-1} = A.$ ✓

14. Let $\lambda_1, \dots, \lambda_n \in \mathbb{C}$. Prove that
    $$\det \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix} = \prod_{1 \leq i < j \leq n} (\lambda_j - \lambda_i).$$

    Check Tut 3 from 2021 (bit.ly/ma-106)

15. Let A be an $n \times n$ matrix satisfying $A^2 = A$. Suppose that A is neither the zero matrix nor the identity matrix.
    Choose the correct option(s).
    - (a) A must be invertible. ✗
    - [✓ (b)] A cannot be invertible.
    - [✓ (c)] The only possible eigenvalues of A are 0 and 1.
    - (d) The null space and column space of A have a nonzero vector in common.

    If A is inv, then $A^2 = A \Rightarrow A = I$. A contradiction.

    Suppose $v \in \mathcal{N}(A) \cap \mathcal{C}(A)$.

    $Av = 0$.

    $v = Aw$

    $0 = Av = A(Aw) = A^2 w = Aw = v$ ✓

    let λ — eval

    $v \to$ evec.

    $Av = \lambda v$, $A^2 v = \lambda^2 v$, $Av = A^2v$ $\Rightarrow \lambda v = \lambda^2 v$, $v \neq 0$

    $\Rightarrow \lambda = \lambda^2$

    $\Rightarrow \lambda \in \{0, 1\}$.

2

If A is similar to O, then A is itself.

$\boxed{A = P^{-1}OP = O.}$

$(P^{-1}BP)^2 = P^{-1}B^2P$

$A \sim B$

$\Updownarrow$

$A - 2I \sim B - 2I$

$\Downarrow$

$(A - 2I)^2 \sim (B - 2I)^2$

$(A-2I)^2 = 0 \qquad (B-2I)^2 \neq 0$

Contradiction.

↳ let λ — eval

$v \to$ evec.

$Av = \lambda v$

$A^2 v = \lambda^2 v$ $\Rightarrow$ $\lambda v = \lambda^2 v$, $v \neq 0$

$\Rightarrow \lambda = \lambda^2$

$\Rightarrow \lambda \in \{0, 1\}$.

$B^2 = P\begin{bmatrix} a & & \\ & b & \\ & & c \end{bmatrix}P^{-1} = A.$ ✓

Let $x \in \mathbb{R}^n$. $x = (x - Ax) + Ax.$

↳ 0-eigenspace ↳ 1-eigenspace

$A(x - Ax) = 0$

$A(Ax) = Ax.$

16. Show that if A is an $n \times n$ matrix satisfying $A^2 = A$, then A is diagonalisable. ✓
Conclude that if $A^2 = cA$ for some $c \neq 0$, then too A is diagonalisable. $\to (A/c)^2 = (A/c) \rightsquigarrow A/c$ is diagona. $\Downarrow$ A is . . .

17. Let A be a matrix such that $A^k = O$ for some $k \geq 1$. Show that $I - A$ is invertible.
↳ $I + A + \cdots + A^{k-1}$

18. Let A and B be 4×4 matrices defined by

$$A = \begin{bmatrix} 2 & 3 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 4 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

$\begin{bmatrix} 0\ 3 & 0 \\ 0\ 0 & \\ 0 & 0\ 4 \\ & 0\ 0 \end{bmatrix}$ $\begin{bmatrix} 0\ 4\ 0 & 0 \\ 0\ 0\ 3 & \\ 0\ 0\ 0 & 0 \\ 0\ 0\ 0 & \end{bmatrix}$

Mark the correct option(s).

✓(a) Both A and B have the same characteristic polynomial. $\to (x-2)^4$

✓(b) Both A and B have the same eigenvalues and their geometric multiplicities are also the same. $A - 2I = \begin{bmatrix} 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightsquigarrow$ REF rank $= 2$, nullity $= 2$. Similarly for B.

✓(c) Both A and B have the same eigenvalues and their algebraic multiplicities are also the same.

✗(d) A and B are similar.

19. Consider

$$A = \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 3 & 0 & 0 & 3 \\ 0 & -1 & 0 & 2 \end{bmatrix}, \ \mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \text{ and } \mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 1 \end{bmatrix}.$$

Choose the correct option(s):

(a) $A\mathbf{x} = \mathbf{u}$ has a solution.

(b) $\mathbf{v}$ is in the column space of A. $\Leftrightarrow Ax = v$ has a sol^n.

(c) None of the above.

$\begin{bmatrix} A & | u & | v \end{bmatrix}$ $\xrightarrow{}$ REF

Check!

20. Find the value(s) of k for which the system

$$\begin{aligned} y + 3kz &= 0 \\ x + 2y + 6z &= 2 \\ kx + 2ky + 12z &= -4 \end{aligned}$$

has no solution.

$\left[\begin{array}{ccc|c} 0 & 1 & 3k & 0 \\ 1 & 2 & 6 & 2 \\ k & 2k & 12 & -4 \end{array}\right] \longrightarrow$ REF check.

21. Let A be an $m \times n$ matrix. Let $\mathcal{N}(A)$, $\mathcal{R}(A)$, and $\mathcal{C}(A)$ denote the null space, row space, and column space of A, respectively. Pick the correct option(s).

✗(a) $\dim(\mathcal{N}(A)) = \dim(\mathcal{R}(A))$.

✓(b) $\dim(\mathcal{N}(A)) + \dim(\mathcal{R}(A)) = n$.

✓(c) $\dim(\mathcal{N}(A)) + \dim(\mathcal{C}(A)) = n$.

$\dim(\mathcal{R}(A))$
$= \dim(\mathcal{C}(A))$
$= \text{rank}(A)$

3

$\mathcal{N}(A) \subseteq \mathbb{R}^n$

$\mathcal{C}(A) \subseteq \mathbb{R}^m$

$\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 \\ & 1 \end{pmatrix}$

$A \sim B$

$\Rightarrow A - 2I \sim B - 2I$

$\Rightarrow (A - 2I)^2 \sim (B - 2I)^2$

$\Rightarrow 0 \sim (B - 2I)^2 \neq 0.$

Thus, $A \not\sim B$.

$\mathcal{R}(A) \subseteq \mathbb{R}^n$

$\begin{bmatrix} — a_1 — \\ \vdots \\ — a_m — \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_1 \cdot x \\ \vdots \\ a_m \cdot x \end{bmatrix} = 0$

✓(d) $\mathcal{N}(A)$ and $\mathcal{R}(A)$ are orthogonal.

✗(e) $\mathcal{N}(A)$ and $\mathcal{C}(A)$ are orthogonal.

Recall that subspaces $V, W \subseteq \mathbb{R}^n$ are said to be orthogonal if $\langle v, w \rangle = 0$ for all $v \in V$ and all $w \in W$.

22. Let A be a self-adjoint matrix. Show that if $\langle A\mathbf{x}, \mathbf{x} \rangle = 0$ for all $\mathbf{x} \in \mathbb{C}^n$, then $A = O$. $\longrightarrow$ let λ be an eval of A. v — " — evec. to λ.

$\langle Av, v \rangle = 0$
$\| $
$\lambda \|v\|^2$
$\Rightarrow \lambda = 0.$

A is diago... since A is Herm.

But all evals are 0.

$A \sim O$

$\Rightarrow A = O.$

23. Show that if $\|A\mathbf{x}\| = \|A^*\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{C}^n$, then A is a normal matrix.

$B := AA^* - A^*A.$

TS: $B = 0.$

But $B^* = B.$

Check: $\langle Bx, x \rangle = 0.$

Use #22.

24. Show that if $\|A\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{C}^n$, then A is a unitary matrix. $\to B = A^*A - I.$ ∴

25. Which of the following matrices are diagonalisable?

$$\begin{bmatrix} 5 & -1 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

$(t-4)^2 \leftarrow$ but $g_4 = 1$. ✗ ↺ $(t-3)(t-1)(t+1)(t-1/2)$ ✓ ✗

~~26.~~ Find necessary and sufficient conditions on a, b, c for the following matrix to be diagonalisable:

$$\begin{bmatrix} 2 & a & b \\ 0 & 1 & c \\ 0 & 0 & 2 \end{bmatrix}. \longrightarrow -(t-2)^2(t-1).$$

$\boxed{b + ac = 0}$

Only need to care about 2.

Need: nullity$(A - 2I) = 2$

$\Leftrightarrow$ rank$(A - 2I) = 1$

$A - 2I = \begin{bmatrix} 0 & a & b \\ 0 & -1 & c \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & -1 & c \\ 0 & a & b \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & -1 & c \\ 0 & 0 & b+ac \\ 0 & 0 & 0 \end{bmatrix}$

rank $= 1 \Leftrightarrow \boxed{b + ac = 0}.$

27. Let $\lambda \in \mathbb{C}$. Show that λ is an eigenvalue of A iff $\bar{\lambda}$ is an eigenvalue of A^*.

$(A - \lambda I)x = 0$

check $\|(A^* - \bar{\lambda} I)x\| = 0.$

28. Use Gram-Schmidt to orthonormalise the ordered subset

$$([1\ \ -1\ \ 2\ \ 0]^T, [1\ \ 1\ \ 2\ \ 0]^T, [1\ \ 0\ \ 0\ \ 1])^T$$

and obtain an ordered orthonormal set $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$. Also, find $\mathbf{v}_4$ such that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is an orthonormal basis for $\mathbb{R}^4$. $\longrightarrow$ Tut 5 '21 (bit.ly/ma-106)
Express $[1\ \ -1\ \ 1\ \ -1]^T$ as a linear combination of these four basis vectors.

29. Write down the symmetric matrix A such that the quadric

$$7x^2 + 7y^2 - 2z^2 + 20yz - 20zx - 2xy = 36$$

can be expressed as

$$[x\ \ y\ \ z]\, A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 36.$$

Find an orthogonal matrix Q such that $Q^T A Q$ is diagonal.

$A = \begin{bmatrix} 7 & -1 & -10 \\ -1 & 7 & 10 \\ -10 & 10 & -2 \end{bmatrix}$

$\rightsquigarrow p_A(t) \longrightarrow \lambda_1, \lambda_2, \lambda_3$ (possibly same)

$A - \lambda_1 I \longrightarrow$ find orthonormal basis

Construct P.

30. Let A be an $n \times n$ normal matrix and $\lambda \in \mathbb{C}$.
Show that $A - \lambda I$ is a normal matrix.
Show that if $A\mathbf{x} = \lambda\mathbf{x}$, then $A^*\mathbf{x} = \bar{\lambda}\mathbf{x}$.

(27 is a special case.)

4

B normal $\Leftrightarrow$ $\|Bx\| = \|B^*x\|$

$\|(A - \lambda I)x\| = \|(A^* - \bar{\lambda} I)x\|$
$\|$ $\|$
0 $0 \Rightarrow (A^* - \bar{\lambda} I)x = 0$

$\langle Bx, x \rangle = \langle AA^*x - A^*Ax, x \rangle$
$= \langle AA^*x, x \rangle - \langle A^*Ax, x \rangle$
$= \langle A^*x, A^*x \rangle - \langle Ax, Ax \rangle$
$= \|A^*x\|^2 - \|Ax\|^2 = 0.$

$\langle Ax, y \rangle = \langle x, A^*y \rangle$

$\langle v, w \rangle = w^* v$

$\langle Ax, y \rangle = y^* Ax = (A^*y)^* x = \langle x, A^*y \rangle$

$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = x_1\bar{y}_1 + \cdots + x_n\bar{y}_n.$

$\langle Ax, x \rangle = 0$

Let λ be an eval of A with evec $v \neq 0$.

$\langle Av, v \rangle = 0$

$\langle \lambda v, v \rangle = \lambda \langle v, v \rangle$

$\Rightarrow \lambda \|v\|^2 = 0$

$\Rightarrow \lambda = 0$

$\begin{pmatrix} \lambda & 1 \\ & \lambda \end{pmatrix}$ $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

0 $\qquad 0 \Rightarrow$

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

31. Give an example of a square matrix over $\mathbb{C}$ that is not diagonalisable.

32. Suppose $A \in \mathbb{R}^{3\times 3}$ satisfies $A^3 - 2A^2 = A - 2I$ and has the property that $\det(A) < 0$ and $\operatorname{trace}(A) > 2$.
Find the characteristic polynomial $p(t) = \det(A - tI)$.

It is not $-(t^3 - 2t^2 - t + 2)$.

Possible evals of A are $\{1, -1, 2\}$.

Write $p(t) = -(t - \lambda_1)(t - \lambda_2)(t - \lambda_3)$.

$\lambda_1, \lambda_2, \lambda_3 \in \{1, -1, 2\}$.

$\det(A) < 0 \Rightarrow$ odd # of λ_i are -1.

if all 3, the tr $= -3$. ✗

Only one is -1.

tr $> 2 \Rightarrow$ other are 2 & 2.

$\therefore p(t) = -(t+1)(t-2)^2$. ∎

5

λ is an eval of A.

v ——|— - - -

$Av = \lambda v$

$A^2 v = \lambda^2 v$

$A^3 v = \lambda^3 v$

$A^2 v = A(Av)$
$= A(\lambda v)$
$= \lambda(Av)$
$= \lambda(\lambda v) = \lambda^2 v$.

$(A^3 - 2A^2)v = (A - 2I)v$

$\Rightarrow (\lambda^3 - 2\lambda^2)v = (\lambda - 2)v$

$\Rightarrow \lambda^3 - 2\lambda^2 = \lambda - 2 \quad \Big) v \neq 0$

$\Rightarrow \lambda \in \{1, -1, 2\}$.

eigenspace

$E_\lambda = \{v : Av = \lambda v\}$
$= \{\text{eigenvec. corresp. to } \lambda\} \cup \{0\}$.

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$x = (x - Ax) + Ax$.

$\{v_1, \dots, v_{d_1}\} \circlearrowleft E_0 = N(A - 0I) \rightarrow d_1$

$\{w_1, \dots, w_{d_2}\} \swarrow E_1 = N(A - 1I) \rightarrow d_2$

Any $x \in \mathbb{R}^n$ is a L.C. of elements of E_0 and E_1.

$\Rightarrow \mathbb{R}^n = \operatorname{Span}\{v_1, \dots, v_{d_1}, w_1, \dots, w_{d_2}\}$.

$\Rightarrow d_1 + d_2 = n$.

$d_1 \downarrow$ geom. mult. of 0

$d_2 \searrow$ geo. of 1.