7.7

20 April 2021 21:03

7.7 Which quadric surface does the equation $\overbrace{7x^2 + 7y^2 - 2z^2 + 20yz - 20zx - 2xy}^{Q(x,y,z)} - 36 = 0$ describe? Explain by reducing the quadratic form involved to a diagonal form. Express x, y, z in terms of the new coordinates u, v, w.

$$Q(x, y, z) = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 7 & -1 & -10 \\ -1 & 7 & 10 \\ -10 & 10 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Let $A = \begin{bmatrix} 7 & -1 & -10 \\ -1 & 7 & 10 \\ -10 & 10 & -2 \end{bmatrix}.$

Then, $p_A(t) = -t^3 + 12t^2 + 180t - 1296$
$= -(t-18)(t-6)(t+12).$

Each eigenvalue has geo-mult $= 1$.

- $\lambda = -12$: $\mathcal{N}(A + 12I) = \text{span}\left\{ \underbrace{\frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}}_{\text{orthonormal basis}} \right\}.$

(Do the calculations.
Solve $(A + 12I)x = 0$. Convert $A + 12I$ into REF and solve.)

- $\lambda = 6$: $\mathcal{N}(A - 6I) = \text{span}\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$

- $\lambda = 18$: $\mathcal{N}(A - 18I) = \text{span}\left\{ \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}.$

Define $$U := \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \\ -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & 1/\sqrt{3} \end{bmatrix}.$$

Note that U is orthogonal. Thus, $U^T = U^{-1}$ and

$$U^T A U = D := \begin{bmatrix} -12 & & \\ & 6 & \\ & & 18 \end{bmatrix}.$$

Thus, $A = UDU^T$ and hence,

$$\begin{aligned} Q(x, y, z) &= [x \quad y \quad z] A \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\ &= [x \quad y \quad z] U D U^T \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\ &= \left(U^T \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right)^T D \left(U^T \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right). \end{aligned}$$

Define $$\begin{bmatrix} u \\ v \\ w \end{bmatrix} := U^T \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Then,

$$\begin{aligned} Q(x, y, z) &= [u \quad v \quad w] D \begin{bmatrix} u \\ v \\ w \end{bmatrix} \\ &= -12u^2 + 6v^2 + 18w^2. \end{aligned}$$

The surface becomes

$$-12u^2 + 6v^2 + 18w^2 - 36 = 0$$

or

$$\boxed{-\frac{u^2}{3} + \frac{v^2}{6} + \frac{w^2}{2} = 1.}$$ where $\begin{bmatrix} u \\ v \\ w \end{bmatrix} = U^T \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

There are two positives and one negative. Thus, it is a 1-sheeted hyperboloid.

7.8

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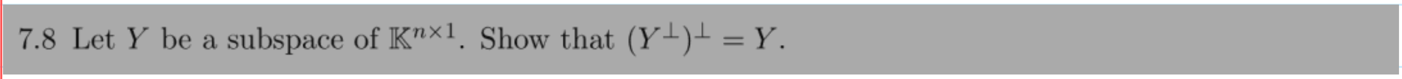

7.8 Let Y be a subspace of $\mathbb{K}^{n\times 1}$. Show that $(Y^{\perp})^{\perp} = Y$.

- **Claim 1.** $Y \subseteq (Y^{\perp})^{\perp}$.

Proof. Let $y \in Y$.
Let $\tilde{y} \in Y^{\perp}$ be arbitrary.

Then, $\langle y, \tilde{y} \rangle = 0$, by definition of $Y^{\perp}$.

Thus, $\langle y, \tilde{y} \rangle = 0 \quad \forall\ \tilde{y} \in Y^{\perp}$.

Thus, $y \in (Y^{\perp})^{\perp}$. ∎

- **Claim 2.** $\dim(Y) = \dim\left((Y^{\perp})^{\perp}\right)$.

Proof. Recall $\mathbb{K}^{n\times 1} = Y \oplus Y^{\perp}$.
Define $P : \mathbb{K}^{n\times 1} \to Y$ to be the orthogonal projection.

Then, $\mathcal{N}(P) = Y^{\perp}$ and $\mathcal{I}(P) = Y$.
$\therefore \dim(V) = \text{rank}(P) + \text{nullity}(P) = \dim(Y) + \dim(Y^{\perp})$.

Thus, $\dim(V) = \dim(Y) + \dim(Y^{\perp})$ — (1) for any subspace $Y \subseteq V$.

Since $Y^{\perp}$ is again a subspace, we have

$$\dim(V) = \dim(Y^{\perp}) + \dim\left((Y^{\perp})^{\perp}\right). \quad \text{— (2)}$$

(1) and (2) give $\dim(Y) = \dim((Y^{\perp})^{\perp})$. $\blacksquare$

- **Claim 3.** $Y = (Y^{\perp})^{\perp}$.

Proof. $Y \subseteq (Y^{\perp})^{\perp}$ is proven and $\dim(Y) = \dim((Y^{\perp})^{\perp})$.
Thus, $Y = (Y^{\perp})^{\perp}$. $\blacksquare$

[If $W \subseteq V \subseteq \mathbb{K}^{n\times 1}$ are subspaces and $\dim(W) = \dim(V)$, then $W = V$.
Proof. Let B be a basis of W.
Then, $|B| = \dim(W) = \dim(V)$. Since B is lin. indep and has "correct" cardinality, it is a basis of V.]

Here, it is important that Y was a subspace to begin with. Otherwise we can only guarantee
$$Y \subseteq (Y^{\perp})^{\perp}.$$

(And also that $\mathbb{K}^{n\times 1}$ is finite dimensional.)

Aliter (Waqar): $Y \subseteq (Y^{\perp})^{\perp}$ is as before.

- Claim : $(Y^{\perp})^{\perp} \subseteq Y$.

Proof. Let $x \in (Y^{\perp})^{\perp}$.
Since Y is a subspace, the projection thm tells us that
$$x = y + \tilde{y} \quad \text{for some } y \in Y,\ \tilde{y} \in Y^{\perp}.$$

Thus, $\underset{\|}{\langle x, \tilde{y}\rangle} = \underset{''}{\langle y, \tilde{y}\rangle} + \langle \tilde{y}, \tilde{y}\rangle$.

Thus, $\underset{\substack{\| \\ 0 \\ \text{since} \\ x \in (y^{\perp})^{\perp}}}{\langle x, \tilde{y} \rangle} = \underset{\substack{\| \\ 0 \\ \text{since} \\ \tilde{y} \in y^{\perp}}}{\langle y, \tilde{y} \rangle} + \langle \tilde{y}, \tilde{y} \rangle.$

$\therefore \quad \|\tilde{y}\|^2 = 0 \quad \text{or} \quad \tilde{y} = 0.$

Thus, $x = y \in Y.$ $\blacksquare$

7.9

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7.9 ①Let $\mathbf{A}$ be a self-adjoint matrix. If $\langle \mathbf{Ax}, \mathbf{x}\rangle = 0$ for all $\mathbf{x} \in \mathbb{K}^{n\times 1}$, then show that $\mathbf{A} = \mathbf{O}$. ②Deduce that if $\|\mathbf{A}^*\mathbf{x}\| = \|\mathbf{Ax}\|$ for all $\mathbf{x} \in \mathbb{K}^{n\times 1}$, then $\mathbf{A}$ is a normal matrix, and ③if $\|\mathbf{Ax}\| = \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{K}^{n\times 1}$, then $\mathbf{A}$ is a unitary matrix.

① Since A is self-adjoint, $\exists\, D \in \mathbb{K}^{n\times n}$ and invertible $U \in \mathbb{K}^{n\times n}$ s.t.
↳ diagonal

$$U^{-1} A U = D.$$

(Since A is self-adjoint, the above is true even if $\mathbb{K} = \mathbb{R}$.)

Claim. $D = O$.

Proof Let λ be a diagonal entry of D.
Then, λ is an eigenvalue of A.
Let $0 \neq u \in \mathbb{K}^{n\times 1}$ be an eigenvector of A corresponding to λ.
Then,

$$\bar{\lambda}\|u\|^2 = \bar{\lambda}\langle u, u\rangle = \langle \lambda u, u\rangle = \langle Au, u\rangle = 0.$$

Thus, $\bar{\lambda} = 0$ or $\lambda = 0$.

$\therefore\ D = O.$ ☒

Thus, $A = UDU^{-1} = O$, as desired. ☒

(All we needed was that A is diagonalisable.)

② $\|Ax\| = \|A^* x\| \quad \forall\, x \in \mathbb{K}^{n\times 1}$. TS: A is normal.

Define $B = AA^* - A^*A$.

Then, $B^* = B$. (Check!)

Note $\langle Bx, x\rangle = \langle (AA^* - A^*A)x, x\rangle$

$= \langle AA^*x, x\rangle + \overline{(-1)}\,\langle A^*Ax, x\rangle$

$= \langle A^*x, A^*x\rangle - \langle Ax, Ax\rangle$

$= \|A^*x\|^2 - \|Ax\|^2 = 0.$

By ①, $B = 0$. $\therefore$ $A^*A = AA^*$. ∎

③ Given: $\|Ax\| = \|x\| \quad \forall x \in \mathbb{K}^{n\times 1}$.

TS: A is unitary.

Proof. $B = A^*A - I$. Then, $B^* = B$.

$\langle Bx, x\rangle = \langle A^*Ax - x, x\rangle$

$= \langle A^*Ax, x\rangle - \langle x, x\rangle$

$= \langle Ax, Ax\rangle - \langle x, x\rangle$

$= \|Ax\|^2 - \|x\|^2 = 0.$

By ①, $B = 0$ or $A^*A = I$. ∎

7.10

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7.10 Let E be a nonempty subset of $\mathbb{K}^{n\times 1}$.

(i) If E is not closed, then show that there is $\mathbf{x} \in \mathbb{K}^{n\times 1}$ such that no best approximation to $\mathbf{x}$ exists from E.

(ii) If E is convex, then show that for every $\mathbf{x} \in \mathbb{K}^{n\times 1}$, there is at most one best approximation to $\mathbf{x}$ from E.

(i) E is not closed

$\Leftrightarrow$ There is a sequence (x_n) in E such that (x_n) converges to $x \in \mathbb{K}^{n\times 1} \setminus E$.

TS: $\exists\, x \in \mathbb{K}^{n\times 1}$ s.t. there is no best approx to x from E.

Proof. Take $\overset{\in \mathbb{K}^{n\times 1}\setminus E}{x}$ s.t. $\exists\,(x_n)$ in E which converges to x.
(By defn of E not being closed)

For sake of contradiction, assume $a \in E$ is a best approximation. Then $a \neq x$ since $x \notin E$ and hence

$$\varepsilon := \frac{\|x - a\|}{2} > 0.$$

By defn of convergence, $\exists N \in \mathbb{N}$ s.t.

$$\|x_n - x\| < \varepsilon \quad \forall\, n \geq N.$$

Thus, $\|x_N - x\| < \|x - a\|$. This contradicts the fact that a is a best approximation.

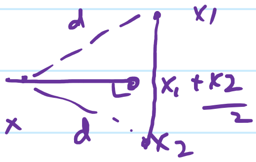
Thus, we are done. ∎

(ii) <u>Recall</u>:

$E \subseteq K^{n \times 1}$ is convex if

$\forall\, x, y \in E$ and $\forall\, t \in [0, 1]$: $t x + (1-t) y \in E$.

Suppose $x \in K^{n\times 1}$ has two distinct best approximations $x_1, x_2 \in E$.



$v_1 := x - x_1$, $v_2 := x - x_2$.

Let $\bar{x} = \frac{x_1 + x_2}{2}$. Note $\|v_1\| = \|v_2\| =: d$ (say).

$$\|x - \bar{x}\| = \left\| x - \left(\frac{x_1+x_2}{2}\right)\right\| = \frac{1}{2}\|v_1 + v_2\|.$$

$$\|v_1 + v_2\|^2 = 2\|v_1\|^2 + 2\|v_2\|^2 - \|v_1 - v_2\|^2$$

(Parallelogram rule)

$$= 4d^2 - \|v_1 - v_2\|^2$$

Since $v_1 \neq v_2$

$$< 4d^2$$

$\Rightarrow \|v_1 + v_2\| < 2d$ or $\frac{1}{2}\|v_1 + v_2\| < d$.

$$\frac{1}{2}\|v_1 + v_2\| = \|x - \bar{x}\|$$

$$\|x - \bar{x}\| < \|x - x_1\|.$$

But $\bar{x} \in E$ since E is convex. This contradicts that x_1 was a best approximation of x from E.

7.11

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7.11 Find $\mathbf{x} := [x_1, x_2]^\top \in \mathbb{R}^{2\times 1}$ such that the straight line $t = x_1 + x_2 s$ fits the data points $(-1, 2)$, $(0, 0)$, $(1, -3)$ and $(2, -5)$ best in the 'least squares' sense.

$$A := \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad b := \begin{bmatrix} 2 \\ 0 \\ -3 \\ -5 \end{bmatrix}.$$

Let $x = [x_1 \quad x_2]^T$. Want to minimise $\|Ax - b\|$.

Thus, we the best approximation a of b from $\mathscr{C}(A)$.

Then, we find $x \in \mathbb{R}^{2\times 1}$ s.t. $Ax = a$.

Step 1. Getting an orthonormal basis (o.n.b.) for $\mathscr{C}(A)$.

Applying GSOP on columns.

$$y_1 := [1 \quad 1 \quad 1 \quad 1]^T.$$

$$y_2 := [-1 \quad 0 \quad 1 \quad 2] - \frac{2}{4}[1 \quad 1 \quad 1 \quad 1]^T$$

$$= [-3/2 \quad -1/2 \quad 1/2 \quad 3/2]$$

$$\therefore \; u_1 = \frac{1}{2}[1 \quad 1 \quad 1 \quad 1]^T$$

$$\text{and} \quad u_2 = \frac{1}{\sqrt{20}}[-3 \quad -1 \quad 1 \quad 3]^T.$$

$\therefore \{u_1, u_2\}$ is an o.n.b. for $\mathscr{C}(A)$.

Step 2. Finding the best approximation.

This is $a = \langle u_1, b\rangle u_1 + \langle u_2, b\rangle u_2$

$$= (-3)u_1 + \frac{-24}{\sqrt{20}} u_2$$

$$= -\frac{3}{2}\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}^T - \frac{\overset{6}{\cancel{24}}}{\underset{5}{\cancel{20}}}\begin{bmatrix}-3 & -1 & 1 & 3\end{bmatrix}^T$$

$$= \begin{bmatrix}\frac{21}{10} & -\frac{3}{10} & -\frac{27}{10} & -\frac{51}{10}\end{bmatrix}^T.$$

Thus, $a = \frac{3}{10}\begin{bmatrix}7 & -1 & 9 & -17\end{bmatrix}^T$ is the best approximation of b from $\mathcal{C}(A)$.

<u>Step 3</u>: Getting $x = [x_1 \quad x_2]^T$ such that $Ax = a$.

$$\left[\begin{array}{cc|c} 1 & -1 & 21/10 \\ 1 & 0 & -3/10 \\ 1 & 1 & -27/10 \\ 1 & 2 & -51/10 \end{array}\right]$$

(Note that we already know this system is consistent since $a \in \mathcal{C}(A)$.)

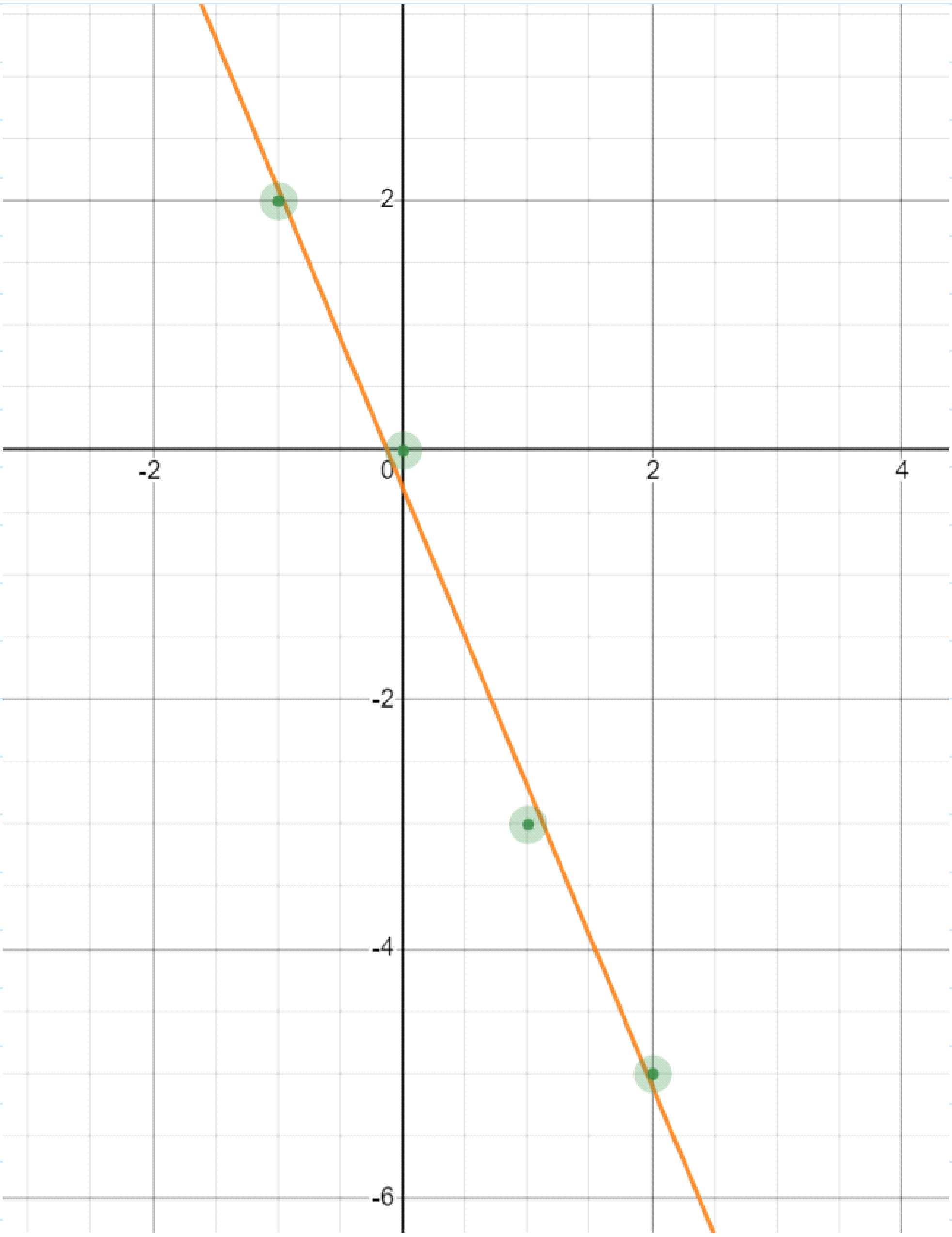
Second equation gives $x_1 = -3/10$.

First equation gives $x_1 - x_2 = \frac{21}{10}$

and hence, $x_2 = x_1 - \frac{21}{10} = -\frac{24}{10}$.

Thus, the desired line is $t = -\frac{3}{10} - \frac{24}{10}s$

$$= -\frac{3}{10} - \frac{12}{5}s.$$


2
-2
-4
-6
-2
0
2
4

7.12

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7.12. Let $Q(x_1,\dots,x_n) := \sum_{j=1}^{n}\sum_{k=1}^{n}\alpha_{jk}x_k\overline{x}_j$, where $\alpha_{jk}\in\mathbb{C}$, be a **complex quadratic form**. If the complex quadratic form $Q(x_1,\dots,x_n)$ takes values in $\mathbb{R}$ for all $[x_1 \ \cdots \ x_n]^{\mathsf{T}}\in\mathbb{C}^{n\times 1}$, then show that there is a unique self-adjoint matrix $\mathbf{A}$ such that

$$Q(x_1,\dots,x_n)=\mathbf{x}^*\mathbf{A}\,\mathbf{x} \quad \text{for all } \mathbf{x}:=[x_1 \ \cdots \ x_n]^{\mathsf{T}}\in\mathbb{C}^{n\times 1}.$$

- $Q(x_1, \dots, x_n) = x^* A x$, where

$$A := \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{bmatrix}.$$

- Claim. $A = A^*$.

Proof. Write $A = B + iC$ where $B, C \in \mathbb{C}^{n\times n}$ are self-adjoint.

$$\left[B = \frac{1}{2}(A + A^*), \quad C = \frac{1}{2i}(A - A^*). \right]$$

$$\langle Ax, x\rangle = \langle Bx, x\rangle - i\,\langle Cx, x\rangle \quad \forall x \in \mathbb{C}^{n\times 1}.$$

Since B, C are self-adjoint, we get

$$\langle Bx, x\rangle, \ \langle Cx, x\rangle \in \mathbb{R} \quad \forall x \in \mathbb{C}^{n\times 1}.$$

(*)

Since $\langle Ax, x\rangle \in \mathbb{R} \quad \forall x \in \mathbb{C}^{n\times 1}$, (given)

we get $\langle Cx, x\rangle = 0 \quad \forall x \in \mathbb{C}^{n\times 1}.$

Thus, $C = 0$. (By 7.9.)

$\therefore$ $A = B$ and B is self-adjoint.
Thus, A is self-adjoint. ∎

Proof of $(*)$ TS: $\langle Bx, x\rangle \in \mathbb{R}$ $\forall x \in \mathbb{C}^{n\times 1}$.

Since $B = B^*$, $\exists$ an orthonormal eigenbasis $\{u_1, \dots, u_n\}$ where the corresp. λ_i are real.

Now, given $x \in \mathbb{C}^{n\times 1}$, write

$$x = a_1 u_1 + \cdots + a_n u_n = \sum_{i=1}^{n} a_i u_i. \quad (a_i \in \mathbb{C}.)$$

Then,

$$\langle Bx, x\rangle = \left\langle \sum_{i=1}^{n} \lambda_i a_i u_i, \sum_{i=1}^{n} a_i u_i \right\rangle$$

$$= \sum_{i=1}^{n} \overline{\lambda_i} |a_i|^2 \quad \lambda_i \in \mathbb{R}$$

$$= \sum_{i=1}^{n} \lambda_i |a_i|^2 \in \mathbb{R}.$$

$\lambda_i, |a_i| \in \mathbb{R}$

- Uniqueness of A.

Suppose A' is self-adjoint s.t. $x^* A x = x^* A' x$ $\forall x \in \mathbb{C}^{n\times 1}$.
Put $B = A - A'$. This is also self adjoint and

$$x^* B x = 0 \quad \forall x \in \mathbb{C}^{n\times 1}.$$

$$\|$$

$$\langle Bx, x\rangle$$

Conclude as before.

7.13

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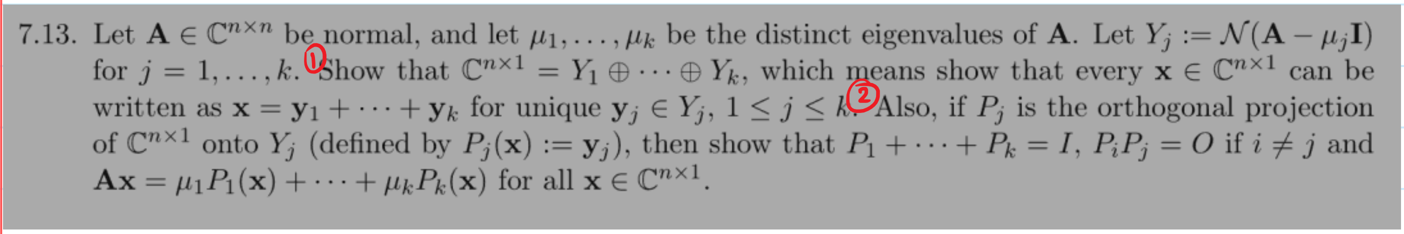

7.13. Let $\mathbf{A} \in \mathbb{C}^{n\times n}$ be normal, and let $\mu_1, \ldots, \mu_k$ be the distinct eigenvalues of $\mathbf{A}$. Let $Y_j := \mathcal{N}(\mathbf{A} - \mu_j \mathbf{I})$ for $j = 1, \ldots, k$. ① Show that $\mathbb{C}^{n\times 1} = Y_1 \oplus \cdots \oplus Y_k$, which means show that every $\mathbf{x} \in \mathbb{C}^{n\times 1}$ can be written as $\mathbf{x} = \mathbf{y}_1 + \cdots + \mathbf{y}_k$ for unique $\mathbf{y}_j \in Y_j$, $1 \le j \le k$. ② Also, if P_j is the orthogonal projection of $\mathbb{C}^{n\times 1}$ onto Y_j (defined by $P_j(\mathbf{x}) := \mathbf{y}_j$), then show that $P_1 + \cdots + P_k = I$, $P_iP_j = O$ if $i \neq j$ and $\mathbf{A}\mathbf{x} = \mu_1 P_1(\mathbf{x}) + \cdots + \mu_k P_k(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{C}^{n\times 1}$.

Let g_i = geo-mult. of μ_i.

Then, we have an orthonormal eigen basis:

$$B = \{ \underbrace{u_1^{(1)}, \ldots, u_1^{(g_1)}}_{\text{orthonormal eigenbasis of } Y_1}, \underbrace{u_2^{(1)}, \ldots, u_2^{(g_2)}}_{\text{of } Y_2}, \ldots, \underbrace{u_k^{(1)}, \ldots, u_k^{(g_k)}}_{\text{of } Y_k} \}.$$

① Given $x \in \mathbb{C}^{n\times 1}$, define

$$y_1 = \langle u_1^{(1)}, x\rangle u_1^{(1)} + \cdots + \langle u_1^{(g_1)}, x\rangle u_1^{(g_1)},$$

$$\vdots$$

$$y_k = \langle u_k^{(1)}, x\rangle u_k^{(1)} + \cdots + \langle u_k^{(g_k)}, x\rangle u_k^{(g_k)}.$$

Since B is an o.n.b., we have

$$x = \sum_{u \in B} \langle u, x\rangle u$$

$$x = y_1 + \cdots + y_k. \qquad \text{—— (1)}$$

Thus, each x can be written as a sum of vectors in Y_i.

Uniqueness Suppose $y_1' \in Y_1, \ldots, y_k' \in Y_k$ were s.t.

$$\underset{\overset{\|}{y_1 + \cdots + y_k}}{x} = y_1' + \cdots + y_k'.$$

Then, $$\overset{Y_1}{\overset{\downarrow}{(y_1 - y_1')}} + \cdots + \overset{Y_k}{\overset{\downarrow}{(y_k - y_k')}} = 0.$$

eigenvectors corresp. to distinct eigenvalues

$$\therefore \quad y_1 - y_1' = \cdots = y_k - y_k' = 0$$

$\therefore \; y_i = y_i' \;\; \forall\, i.$ This proves uniqueness. $\square$

② By eq (1), we have $I = P_1 + \cdots + P_k$ since

$$I(x) = x = y_1 + \cdots + y_k$$

$$= P_1(x) + \cdots + P_k(x).$$

$$P_i P_j = 0 \quad \forall\; i \neq j.$$

Let $x \in \mathbb{C}^{n \times 1}$. Then, $P_j(x) \in Y_j$.

Let $y_j = P_j(x)$.

$$\therefore \quad P_j(x) = 0 + \cdots + 0 + y_j + 0 + \cdots + 0$$

Since $i \neq j$, $P_i(P_j(x)) = 0$.

Since x was arbitrary, $P_i P_j = 0$.

Given $x \in \mathbb{C}^{n \times n}$, write $x = \sum_{i=1}^{n} y_i$ for $y_i \in Y_i$.

Then, $Ax = \sum_{i=1}^{n} A y_i = \sum_{i=1}^{n} \mu_i y_i$

$= \sum_{i=1}^{n} \mu_i \; P_i(x)$.

$\therefore \; Ax = \sum \mu_i P_i(x)$. ∎