5.1

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5.1 Find all eigenvalues, and their geometric as well as algebraic multiplicities of the following matrices. Are they diagonalizable? If so, find invertible $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is a diagonal matrix.

(i) $\mathbf{A} := \begin{bmatrix} 5 & -1 \\ 1 & 3 \end{bmatrix}$, (ii) $\mathbf{A} := \begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$, (iii) $\mathbf{A} := \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$.

(i) $p_A(t) = \det(A - tI) = \det \begin{bmatrix} 5-t & -1 \\ 1 & 3-t \end{bmatrix}$

$= t^2 - 8t + 16 = (t-4)^2$

Thus, the only eigenvalue of A is 4 with algebraic multiplicity 2

However, $\text{nullity}(A - 4I) = \text{nullity} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$

$= 2 - \text{rank} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$

$= 2 - \text{rank} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = 2 - 1 = 1$

Thus, geometric-multiplicity of $4 \neq$ alg-mult(4)

∴ A is not diagonalisable

(ii) $p_A(t) = \det(A - tI)$

$= \det \begin{bmatrix} 3-t & 2 & 1 & 0 \\ 0 & 1-t & 0 & 1 \\ 0 & 2 & -1-t & 0 \\ 0 & 0 & 0 & \frac{1}{2}-t \end{bmatrix}$

$= (3-t) \det \begin{bmatrix} 1-t & 0 & 1 \\ 2 & -1-t & 0 \\ 0 & 0 & \frac{1}{2}-t \end{bmatrix}$

$$= (3-t)\left[(1-t)(t+1)(t-1/2)\right]$$

$$= (t+1)(t-1/2)(t-1)(t-3)$$

Thus, the eigenvalues are $-1, 1/2, 1, 3$, all with algebraic multiplicity 1. Since they geometric multiplicity of each is at least 1, we see that the sum of geo mult s is 4. Thus, A is diagonalisable

To find P we find a basis of eigenspace for each eigenvalue

- $\lambda = -1$ $\quad A + I = \begin{bmatrix} 4 & 2 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3/2 \end{bmatrix}.$

Do two EROs to get an REF of A+I as

$$\begin{bmatrix} 4 & 2 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus, an eigenbasis is $\{[1 \ \ 0 \ \ -4 \ \ 0]^T\}$.

- $\lambda = 1/2$ $\quad A - \frac{1}{2} I = \begin{bmatrix} 5/2 & 2 & 1 & 0 \\ 0 & 1/2 & 0 & 1 \\ 0 & 2 & -3/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$

$\downarrow$ REF

$$\begin{bmatrix} 5/2 & 2 & 1 & 0 \\ 0 & 1/2 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1/2 & 0 & 1 \\ 0 & 0 & -3/2 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, $\{[8 \quad -6 \quad -8 \quad 3]^T\}$

- $\lambda = 1$ $\quad A - I = \begin{bmatrix} 2 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & -1/2 \end{bmatrix}$

$$\xrightarrow{\text{REF}} \begin{bmatrix} 2 & 2 & 1 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Eigenbasis $\{[-3 \quad 2 \quad 2 \quad 0]^T\}$

- $\lambda = 3$ $\quad A - 3I = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 0 & -2 & 0 & 1 \\ 0 & 2 & -4 & 0 \\ 0 & 0 & 0 & -5/2 \end{bmatrix}$

$$\xrightarrow{\text{REF}} = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Eigenbasis $\{[1 \quad 0 \quad 0 \quad 0]^T\}$

Thus, $P = \begin{bmatrix} 1 & -3 & 8 & 1 \\ 0 & 2 & -6 & 0 \\ 0 & 2 & -8 & -4 \\ 0 & 0 & 3 & 0 \end{bmatrix}$ works

(iii) $p_A(t) = (t-2)^3$ but nullity $(A - 2I) = 1$

Thus, geo-mult (2) < alg-mult (2) and thus, A is not diagonalisable

5.2

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5.2 Let $\mathbf{A} := \begin{bmatrix} 2 & a & b \\ 0 & 1 & c \\ 0 & 0 & 2 \end{bmatrix}$. Find a necessary and sufficient condition on a, b, c for $\mathbf{A}$ to be diagonalizable.

(I) $P_A(t) = \det(A - tI)$
$= (2-t)^2(1-t)$

(II) Thus, the eigenvalues are 1 and 2 with $AM(1) = 1$ and $AM(2) = 2$

(III) $1 \le GM(1) \le AM(1) = 1$ Thus, $GM(1) = 1$.

Thus, A is diagonalisable $\Leftrightarrow$ $G(1) + GM(2) = 3$
$\Leftrightarrow$ $GM(2) = 2$

(IV) Now, we calculate $GM(2) = \text{nullity}(A - 2I)$
$= 3 - \text{rank}(A - 2I)$

$$A - 2I = \begin{bmatrix} 0 & a & b \\ 0 & -1 & c \\ 0 & 0 & 0 \end{bmatrix}$$

We compute an REF of $A - 2I$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 0 & -1 & c \\ 0 & a & b \\ 0 & 0 & 0 \end{bmatrix}$$

$R_2 + aR_1$

$$\begin{bmatrix} 0 & \boxed{-1} & c \\ 0 & 0 & b + ac \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, $\text{rank}(A - 2I) = 1 \iff b + ac = 0$

$$\Updownarrow$$

$\text{nullity}(A - 2I) = 2$

$$\Updownarrow$$

$GM(2) = 2$

Thus, A is diagonalisable $\iff \boxed{b + ac = 0}$

nec + suff condition

5.3

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5.3 Let $k \in \mathbb{N}$ and

$$\mathbf{A}_k := \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & \cdots & \cdots & 0 \\ 1 & 0 & 0 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & -1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 0 & 0 & 0 & -1 \\ 0 & \cdots & \cdots & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \in \mathbb{K}^{2k \times 2k},$$

that is, $\mathbf{A}$ has all diagonal entries 0, the subdiagonal entries are $1, 0, 1, 0 \ldots, 1, 0$, and the superdiagonal entries are $-1, 0, -1, 0, \ldots, -1, 0$. Find the characteristic polynomial of $\mathbf{A}$, all eigenvalues of $\mathbf{A}$, and their algebraic as well as geometric multiplicities.

Fact (Not using this w/o proof.)

square

$$M = \begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_r \end{bmatrix}$$

$A_i \to$ square matrices

Then, $\det(M) = \det(A_1) \cdots \det(A_r)$

Claim $\det(A_k - tI) = (t^2+1)^k \quad \forall\, k \in \mathbb{N}$

Proof For $k=1$, we have $\det(A_1 - tI) = \det \begin{bmatrix} -t & -1 \\ 1 & -t \end{bmatrix}$

$= t^2 + 1$

Assume true for some k, where $k \geq 1$

Now,

$$\det(A_{k+1} - tI) = \begin{bmatrix} -t & -1 & & & & & \\ 1 & -t & & & & & \\ & & -t & -1 & & & \\ & & 1 & -t & & & \\ & & & & \ddots & & \\ & & & & & -t & -1 \\ & & & & & 1 & -t \end{bmatrix}$$

Expand along first row to get

$$\det(A_{k+1} - tI) = -t \det\begin{bmatrix} -t & & & & & \\ & -t & -1 & & & \\ & 1 & -t & & & \\ & & & \ddots & & \\ & & & & -t & -1 \\ & & & & 1 & -t \end{bmatrix}$$

$$-(-1)\det\begin{bmatrix} 1 & & & & & \\ & -t & -1 & & & \\ & 1 & -t & & & \\ & & & \ddots & & \\ & & & & -t & -1 \\ & & & & 1 & -t \end{bmatrix}$$

Expand both along first row to get:

$$\det(A_{k+1} - tI) = (-t)^2 \det(A_k) + 1 \det(A_k)$$
$$= (t^2+1)\det(A_k)$$

By induction, the claim follows ☒

Thus, $p_A(t) = (t^2+1)^k$

- If $\mathbb{K} = \mathbb{R}$, then there are no eigenvalues (or eigenvectors)
- Now, assume $\mathbb{K} = \mathbb{C}$

Then, $p_A(t) = (t+2)^k (t-2)^k$

Thus, the eigenvalues are $2 \to AM = k,$
$-2 \to AM = k$

$$\underbrace{\begin{bmatrix} 1 \\ i \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}_{2k\times 1}}_{v_1}, \quad \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ i \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{2k\times 1}}_{v_2}, \quad \ldots, \quad \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \\ i \end{bmatrix}_{2k\times 1}}_{v_k}$$

are k lin indep eigenvectors of A_k corresponding to i

Thus, $GM(i) \geq k$. But $AM(i) = k$

$$GM(i) = k$$

Similarly, $\bar{v}_1, \ldots, \bar{v}_k$ are lin indep e-vectors of A_k corresp to $-i$. Since $AM(-i) = k$, we get $GM(-i) = k$

Thus,

eigenvalue	AM	GM
i	k	k
$-i$	k	k

In particular, A_k is diagonalisable for all $k \in \mathbb{N}$.

5.4

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5.4 Let $\lambda \in \mathbb{K}$.(i) Show that λ is an eigenvalue of $\mathbf{A}$ if and only if $\bar{\lambda}$ is an eigenvalue of $\mathbf{A}^*$,(ii) but their eigenvectors can be very different.

Recall $A^* = \overline{(A^T)}$

(i) Note $(A - \lambda I)^* = A^* - \bar{\lambda} I^* = A^* - \bar{\lambda} I$

Thus, $\det(A - \lambda I) = 0 \Leftrightarrow \det(A^* - \bar{\lambda} I) = 0$

$\left(\det(A^*) = \overline{\det(A)}\right)$

Thus, λ is an eval of $A \Leftrightarrow \det(A - \lambda I) = 0$

$\Leftrightarrow \det(A^* - \bar{\lambda} I) = 0$

$\Leftrightarrow \bar{\lambda}$ is an eval of A^*

(ii) Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ Then, A has 1 as an eigenvalue and e_1 is an e-vector of A corresp to 1

But $A^* = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ does not have e_1 as eigenvector.

5.5

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5.5 Let $\mathbf{A} \in \mathbb{K}^{n\times n}$. Show that 0 is an eigenvalue of $\mathbf{A}$ if and only if 0 is an eigenvalue of $\mathbf{A}^*\mathbf{A}$, and its geometric multiplicity is the same. Deduce rank $\mathbf{A}^*\mathbf{A} = $ rank $\mathbf{A}$.

Claim $\mathcal{N}(A^*A) = \mathcal{N}(A)$ ($\mathcal{N}(BA) \supseteq \mathcal{N}(A)$, in general)

Proof ($\subseteq$) Let $v \in \mathcal{N}(A^* A)$

Then, $A^* A v = 0$

Thus, $v^* A^* A v = 0$

Hence, $\langle Av, Av \rangle = 0$ or $\|Av\| = 0$

This is possible only if $Av = 0$ $v \in \mathcal{N}(A)$

($\supseteq$) Let $v \in \mathcal{N}(A)$

Then, $Av = 0$ and thus, $A^* A v = 0$

Hence, $v \in \mathcal{N}(A^*A)$

(i) 0 is an e-val of A $\iff \mathcal{N}(A) = \mathcal{N}(A - 0I) \neq \{0\}$

$\iff \mathcal{N}(A^* A) \neq \{0\}$ By claim

$\iff 0$ is an e-val of A^*A

(ii) GM of 0 of A $=$ nullity $(A) = $ nullity $(A^*A) = $ GM of 0 of A^*A

By claim

(iii) By rank-nullity, rank$(A^*A) = n -$ nullity $(A^*A) = n -$ nullity $(A) = $ rank(A).

By claim

5.6

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5.6 Let $\mathbf{A} := \begin{bmatrix} 2 & i & 1+i \\ -i & 3 & 1 \\ 1-i & -1 & 8 \end{bmatrix}$. Show that no eigenvalue of $\mathbf{A}$ is away from one of the diagonal entries of $\mathbf{A}$ by more than $1+\sqrt{2}$.

(Gergschorin?)

Let λ be an eigenvalue of A Then, $A - \lambda I$ is not invertible Thus, it is not strictly diagonally dominant (Q5 7)

Thus, one of the following is true. (Possibly two or more may be true as well)

(i) $|2-\lambda| \le |i| + |1+i| = 1+\sqrt{2}$,

(ii) $|3-\lambda| \le |-i| + |1| = 2 \le 1+\sqrt{2}$,

(iii) $|8-\lambda| \le |1-i| + |-1| = 1+\sqrt{2}$

5.7

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5.7 A square matrix $\mathbf{A} \in \mathbb{K}^{n \times n} := [a_{jk}]$ is called **strictly diagonally dominant** if $|a_{jj}| > \sum_{k \neq j} |a_{jk}|$ for each $j = 1, \ldots, n$. If $\mathbf{A}$ strictly diagonally dominant, show that $\mathbf{A}$ is invertible.

Suppose A is not invertible. We show that A is not strictly diagonally dominant

Since A is not invertible, $\exists\, x \in \mathbb{K}^{n \times 1}$ s.t. $Ax = 0$ but $x \neq 0$

$x = [x_1 \quad x_2 \quad \ldots \quad x_n]^T$ for $x_1, \ldots, x_n \in \mathbb{K}$

Choose d s.t. $|x_d| = \max\{|x_1|, \ldots, |x_n|\}$

Then, $|x_d| > 0$ since $x \neq 0$

Now, $Ax = 0$. Comparing the d^{th} entry, we get

$$a_{d1} x_1 + \ldots + a_{dd} x_d + \ldots + a_{dn} x_n = 0$$

$$\Rightarrow a_{dd} x_d = -\sum_{k \neq d} a_{dk} x_k$$

since $|x_d| \geq |x_k| \ \forall k$

$$\Rightarrow |a_{dd} x_d| \leq \sum_{k \neq d} |a_{dk}| |x_k| \leq \sum_{k \neq d} |a_{dk}| |x_d|$$

cancelling $|x_d| > 0$

$$\Rightarrow |a_{dd}| \leq \sum_{k \neq d} |a_{dk}|$$

Thus, A is not strictly diagonally dominant

5.8

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$\|x\|_\infty = \max\{|x_1|, \ldots, |x_n|\}$

$\max\{\|Ax\|_\infty \quad \|x\|_\infty = 1\}$

$=$

5.8 Let $\mathbf{A} \in \mathbb{K}^{n\times n}$. Define $\alpha_2 := \max\{\|\mathbf{Ax}\| : \|\mathbf{x}\| = 1\}$, $\alpha_\infty := \max\left\{\sum_{k=1}^{n} |a_{jk}| : j = 1, \ldots, n\right\}$ and $\alpha_1 := \max\left\{\sum_{j=1}^{n} |a_{jk}| : k = 1, \ldots, n\right\}$, where $\mathbf{A} := [a_{jk}]$. Show that $|\lambda| \le \min\{\alpha_2, \alpha_\infty, \alpha_1\}$ for every eigenvalue λ.

$\shortparallel$

$\max\{\|Ax\|_1 \quad \|x\|_1 = 1\}$ $\quad \|x\|_1 = |x_1| + \quad + |x_n|$

works for any matrix norm

- α_2

  Let $v \neq 0$ be an eigenvector of A corresp to λ

  Put $u = \dfrac{v}{\|v\|}$ Then, $\|u\| = 1$ and $Au = \lambda u$

  (Check!)

  Thus, $|\lambda| = \|\lambda u\| = \|Au\| \le \alpha_2$

- α_1

  As in (Q 5 6), $\exists\, d$ s t

  $$|a_{dd} - \lambda| \le \sum_{k \neq d} |a_{dk}|$$

  But $|\lambda| - |a_{dd}| \le |\lambda - a_{dd}| \le \sum_{k\neq d} |a_{dk}|$

  $$|\lambda| \le \sum_{k=1}^{n} |a_{dk}| \le \alpha_1$$

- α_∞

  A and A^T have same eigenvalues Conclude.

Thus, $|\lambda| \le \min\{\alpha_1, \alpha_2, \alpha_\infty\}$

(Matrix norm If $\|\cdot\|$ is a MATRIX norm and $A \in \mathbb{K}^{n\times n}$ with λ an eval of A, then $|\lambda| \le \|A\|$

Each of α_1, α_2, α_∞ is actually a matrix norm (Induced matrix norm))

[PURPLE can be ignored]

5.9

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5.9 Let $\mathbf{x}, \mathbf{y} \in \mathbb{K}^{n\times 1}$. Prove the **parallelogram law**: $\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2 = 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2$. In case $\mathbf{x}$ and $\mathbf{y}$ are both nonzero, prove the **cosine law**, which says that $\|\mathbf{x}-\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\|\mathbf{x}\|\|\mathbf{y}\|\cos\theta$, where the angle $\theta \in [0, \pi]$ between nonzero $\mathbf{x}$ and $\mathbf{y}$ is defined to be $\cos^{-1}(\Re\langle \mathbf{x}, \mathbf{y}\rangle / \|\mathbf{x}\|\|\mathbf{y}\|)$.

(i) Parallelogram

linearity in 2nd var

$$\begin{aligned}
\|x+y\|^2 + \|x-y\|^2 &= \langle x+y, x+y\rangle + \langle x-y, x-y\rangle \\
&= \langle x+y, x\rangle + \langle x+y, y\rangle + \langle x-y, x\rangle - \langle x-y, y\rangle \\
&= \langle x+y, x\rangle + \bar{1}\langle x-y, x\rangle + \langle x+y, y\rangle - \bar{1}\langle x-y, y\rangle \\
&= \langle 2x, x\rangle + \langle 2y, y\rangle \\
&= \bar{2}\langle x, x\rangle + \bar{2}\langle y, y\rangle \\
&= 2\|x\|^2 + 2\|y\|^2
\end{aligned}$$

conjugate lin in 1st var

Conj lin in 1st var

(ii) Cosine:

$$\begin{aligned}
\|x-y\|^2 = \langle x-y, x-y\rangle &= \langle x-y, x\rangle - \langle x-y, y\rangle \\
&= \langle x, x\rangle - \langle y, x\rangle - \langle x, y\rangle + \langle y, y\rangle \\
&= \|x\|^2 - \overline{\langle x, y\rangle} - \langle x, y\rangle + \|y\|^2 \\
&= \|x\|^2 - 2\mathrm{Re}\langle x, y\rangle + \|y\|^2 \\
&= \|x\|^2 - 2\|x\|\|y\|\,\mathrm{Re}\left(\frac{\langle x, y\rangle}{\|x\|\|y\|}\right) + \|y\|^2 \\
&= \|x\|^2 - 2\|x\|\|y\|\cos(\theta) + \|y\|^2 \quad \blacksquare
\end{aligned}$$

6.1

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6.1 Orthonormalize the following ordered subsets of $\mathbb{K}^{4\times 1}$.

(i) $(\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4)$

(ii) $(\underbrace{\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4}_{= w_1}, \underbrace{-\mathbf{e}_1 + \mathbf{e}_2}_{= w_2}, \underbrace{-\mathbf{e}_1 + \mathbf{e}_3}_{= w_3}, \underbrace{-\mathbf{e}_1 + \mathbf{e}_4}_{= w_4})$.

(ii)

$v_1 = w_1$

$v_2 = w_2 - P_{v_1}(w_2)$

$P_{v_1}(w_2) = \dfrac{\langle v_1, w_2 \rangle}{\|v_1\|^2} v_1 = 0$

$v_2 = w_2$

$v_3 = w_3 - P_{v_2}(w_3) - P_{v_1}(w_3)$

$= w_3 - \dfrac{(+1)}{2} v_2 - 0 = w_3 - \dfrac{v_2}{2} = w_3 - \dfrac{w_2}{2}$

$= -\frac{1}{2} e_1 - \frac{1}{2} e_2 + e_3$

$v_3 = -\frac{1}{2} e_1 - \frac{1}{2} e_2 + e_3$

$v_4 = w_4 - P_{v_3}(w_4) - P_{v_2}(w_4) - \cancelto{0}{P_{v_1}(w_4)}$

$w_4 = -e_1 + e_4$

$v_2 = -e_1 + e_2$

$= w_4 - \dfrac{(1/2)}{(3/2)} v_3 - \dfrac{(1)}{2} v_2$

$= (-e_1 + e_4) - \frac{1}{3}\left(-\frac{1}{2} e_1 - \frac{1}{2} e_2 + e_3\right) - \frac{1}{2}(-e_1 + e_2)$

$= -\frac{1}{3} e_1 - \frac{1}{3} e_2 - \frac{1}{3} e_3 + e_4$

Now, we normalise to get

$$\left(\frac{v_1}{2}, \frac{v_2}{\sqrt{2}}, \frac{v_3}{\sqrt{3/2}}, \frac{v_4}{\sqrt{4/3}} \right)$$

(1) Do it similarly to get (e_1, e_2, e_3, e_4) as answer

6.2

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6.2 Use the Gram-Schmidt Orthogonalization Process to orthonormalize the ordered subset

$$\left(\begin{bmatrix}1 & -1 & 2 & 0\end{bmatrix}^\mathsf{T}, \begin{bmatrix}1 & 1 & 2 & 0\end{bmatrix}^\mathsf{T}, \begin{bmatrix}3 & 0 & 0 & 1\end{bmatrix}^\mathsf{T}\right)$$

and obtain an ordered orthonormal set $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$. Also, find $\mathbf{u}_4$ such that $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4)$ is an ordered orthonormal basis for $\mathbb{K}^{4\times 1}$. Express the vector $\begin{bmatrix}1 & -1 & 1 & -1\end{bmatrix}^\mathsf{T}$ as a linear combination of these four basis vectors.

$$v_1 = w_1 = \begin{bmatrix}1 & -1 & 2 & 0\end{bmatrix}^T$$

$$v_2 = w_2 - P_{v_1}(w_2)$$

$$= w_2 - \frac{4}{6} v_1 = w_2 - \frac{2}{3} v_1$$

$$= \begin{bmatrix}\frac{1}{3} & \frac{5}{3} & \frac{2}{3} & 0\end{bmatrix}^T$$

$$v_3 = w_3 - P_{v_2}(w_3) - P_{v_1}(w_3) \qquad w_3 = \begin{bmatrix}3 & 0 & 0 & 1\end{bmatrix}^T$$

$$= w_3 - \frac{1}{\left(\frac{30}{9}\right)} v_2 - \frac{3}{6} v_1$$

$$= w_3 - \frac{3}{10} v_2 - \frac{1}{2} v_1 = \begin{bmatrix}\frac{12}{5} & 0 & -\frac{6}{5} & 1\end{bmatrix}^T$$

Thus, we get $u_1 = \frac{v_1}{\|v_1\|} = \begin{bmatrix}\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & 0\end{bmatrix}^T,$

$$u_2 = \frac{v_2}{\|v_2\|} = \begin{bmatrix}\frac{1}{\sqrt{30}} & \frac{5}{\sqrt{30}} & \frac{2}{\sqrt{30}} & 0\end{bmatrix}^T,$$

$$u_3 = \frac{v_3}{\|v_3\|} = \begin{bmatrix}\frac{12}{\sqrt{185}} & 0 & -\frac{6}{\sqrt{185}} & \frac{5}{\sqrt{185}}\end{bmatrix}^T$$

Now, we solve for u_4 by first getting a v_4 perpendicular to all u_1, u_2, u_3 by solving

$$\begin{bmatrix}1 & -1 & 2 & 0\\ 1 & 5 & 2 & 0\\ 12 & 0 & -6 & 5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_n\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 6 & 0 \\ 1 & 5 & 2 & 0 & 0 \\ 12 & 0 & -6 & 5 & 0 \end{array}\right]$$

$$\Big\downarrow \begin{array}{l} R_2 - R_1 \\ R_3 - 12R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 \\ 0 & 12 & -30 & 5 & 0 \end{array}\right]$$

$$\Big\downarrow R_3 - 2R_2$$

$$\left[\begin{array}{cccc|c} \boxed{1} & -1 & 2 & 0 & 0 \\ 0 & \boxed{6} & 0 & 0 & 0 \\ 0 & 0 & \boxed{-30} & 5 & 0 \end{array}\right]$$

Thus, $\begin{bmatrix} -1/3 \\ 0 \\ 1/6 \\ 1 \end{bmatrix} = v_4$ is one solution

We get u_4 as $u_4 = \dfrac{v_4}{\|v_4\|} = \dfrac{1}{\sqrt{41}} \begin{bmatrix} -2 \\ 0 \\ 1 \\ 6 \end{bmatrix}$

Now, let $b = [1 \ -1 \ 1 \ -1]^T$

Then, $b = \langle u_1, b \rangle u_1 + \langle u_2, b \rangle u_2 + \langle u_3, b \rangle u_3 + \langle u_4, b \rangle u_4.$

We have
$$\begin{aligned} \langle u_1, b \rangle &= 4/\sqrt{6}, \\ \langle u_2, b \rangle &= -2/\sqrt{30} \\ \langle u_3, b \rangle &= 1/\sqrt{185}, \\ \langle u_4, b \rangle &= -7/\sqrt{41} \end{aligned}$$

Thus, $b = \dfrac{4}{\sqrt{6}} u_1 - \dfrac{2}{\sqrt{30}} u_2 + \dfrac{1}{\sqrt{185}} u_3 - \dfrac{7}{\sqrt{41}} u_4$ ∎

Very well possible that I have made a calculation mistake!