4.3

31 March 2021 10:57

4.3 Find the matrix of the linear transformation $T : \mathbb{R}^{3\times 1} \to \mathbb{R}^{4\times 1}$ defined by $T(\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^\mathsf{T}) := \begin{bmatrix} x_1+x_2 & x_2+x_3 & x_3+x_1 & x_1+x_2+x_3 \end{bmatrix}^\mathsf{T}$ with respect to the ordered bases (i) $E = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ of $\mathbb{R}^{3\times 1}$ and $F = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$ of $\mathbb{R}^{4\times 1}$,

(ii) $E' = (\mathbf{e}_1+\mathbf{e}_2, \mathbf{e}_2+\mathbf{e}_3, \mathbf{e}_3+\mathbf{e}_1)$ of $\mathbb{R}^{3\times 1}$ and $F' = (\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3, \mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4, \mathbf{e}_3+\mathbf{e}_4+\mathbf{e}_1, \mathbf{e}_4+\mathbf{e}_1+\mathbf{e}_2)$ of $\mathbb{R}^{4\times 1}$, first showing that E' is a basis for $\mathbb{R}^{3\times 1}$ and F' is a basis for $\mathbb{R}^{4\times 1}$.

(i) $E = (e_1, e_2, e_3)$; $F = (e_1, e_2, e_3, e_4)$

$$= \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) \qquad \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

$$T(e_1) = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \underline{1} e_1 + \underline{0} e_2 + \underline{1} e_3 + \underline{1} e_4$$

$$T(e_2) = 1e_1 + 1e_2 + 0e_3 + 1e_4$$

$$T(e_3) = 0e_1 + 1e_2 + 1e_3 + 1e_4$$

$$M_F^E(T) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

(ii) $E' = (e_1 + e_2, \quad e_2 + e_3, \quad e_3 + e_1)$

$F' = (e_1 + e_2 + e_3, \quad e_2 + e_3 + e_4, \quad e_3 + e_4 + e_1, \quad e_4 + e_1 + e_2)$

Create the matrices with columns as the given vectors.

$$Q = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

To show E (resp., F) is a basis, it suffices to show that Q (resp., P) is invertible.

For that, we may use GEM to reduce Q and P to an REF.

Aliter: Calculate $\det(P)$ and $\det(Q)$ and show they are non-zero.

Assuming we have done that.

$$T(e_1 + e_2) = T\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix}$$

$$= 0\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$T(e_2 + e_3) = T\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix} = 0\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$+ 1\begin{bmatrix} 1 \\ 1 \\ 0 \\ \end{bmatrix}$$

$$+ 1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$T(e_3 + e_1) = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Thus, $$M^{\mathcal{E}'}_{P'}(T) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

4.4

31 March 2021 13:15

4.4 Let $\mathbf{A} \in \mathbb{R}^{4\times 4}$. Let $\mathbf{P} := \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$. Show that $\mathbf{P}$ is invertible. Find an ordered bases E of $\mathbb{R}^{4\times 1}$ such that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{M}_E^E(T_{\mathbf{A}})$.

Invertibility : Use GEM.

Proposition

Let $\mathbf{A}, \mathbf{B} \in \mathbb{K}^{n\times n}$. Then $\mathbf{A} \sim \mathbf{B}$ if and only if there is an ordered basis E for $\mathbb{K}^{n\times 1}$ such that $\mathbf{B}$ is the matrix of the linear transformation $T_{\mathbf{A}} : \mathbb{K}^{n\times 1} \to \mathbb{K}^{n\times 1}$ with respect to E.

In fact, $\mathbf{B} = \mathbf{P}^{-1}\mathbf{AP}$ if and only if the columns of $\mathbf{P}$ form an ordered basis, say E, for $\mathbb{K}^{n\times 1}$ and $\mathbf{B} = \mathbf{M}_E^E(T_{\mathbf{A}})$.

Thus, $E = \left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right)$

works. E is basis since (i) By 4.3 OR
(ii) the fact that P is invertible

4.5

31 March 2021 10:57

4.5 Let $\lambda \in \mathbb{K}$. Find the geometric multiplicity of the eigenvalue λ of each of the following matrices:

$$\mathbf{A} := \begin{bmatrix} \boxed{\lambda} & 0 & 0 \\ 0 & \boxed{\lambda} & 0 \\ 0 & 0 & \boxed{\lambda} \end{bmatrix}, \mathbf{B} := \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \boxed{\lambda} \end{bmatrix}, \mathbf{C} := \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}.$$

Also, find the eigenspace associated with λ in each case.

Recall : $\text{geo-mult}(\lambda) = \dim(\mathcal{N}(A - \lambda I))$
$= \text{nullity}(A - \lambda I)$.

Can calculate using rank-nullity

(i) $A - \lambda I = \mathbf{0}$

Thus, $\text{rank}(\mathbf{0}) = 0$ and hence,
$\text{nullity}(\mathbf{0}) = 3 - 0 = 3$.

Thus, the geo. mult. is 3.

(ii) $B - \lambda I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

The above is in RCF. $\therefore$ nullity $= 3 - 1 = \underline{\underline{2}}$.

geo. mult.

(iii) $C - \lambda I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Again, in RCF.

$\therefore$ nullity $= 3 - 2 = 1$.

Note that the characteristic poly of all of them is $-(t-\lambda)^3$.

However, the geometric multiplicities are all different.

(In particular, no two of them are similar.)

B and C are not diagonalisable. [In fact, they are in <u>Jordan form</u>.] → not in course.

$$\begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \leftarrow \text{diag.}$$

$$\begin{bmatrix} J_1 & & & 0 \\ & J_2 & & \\ & & \ddots & \\ 0 & & & J_k \end{bmatrix} \leftarrow \text{Jordan}$$

Each J_i is square (different sizes possibly.)

Each J_i looks like $\begin{bmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix}$.

If $\mathbb{K} = \mathbb{C}$, then every matrix is similar to a Jordan matrix.

(Unique up to a permutation of J's.)

4.6

31 March 2021 10:57

Eigenspace = {all eigenvectors} ⊔ {0}.

4.6 Let $\mathbf{A} := \begin{bmatrix} 3 & 0 & 0 \\ -2 & 4 & 2 \\ -2 & 1 & 5 \end{bmatrix}$. ① a Show that 3 is an eigenvalue of $\mathbf{A}$, and b find all eigenvectors of $\mathbf{A}$ corresponding to it. ② a Also, show that $\begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^{\mathsf{T}}$ is an eigenvector of $\mathbf{A}$, and b find the corresponding eigenvalue of $\mathbf{A}$.

① a Suffices to show $\det(A - 3I) = 0$.

$$A - 3I = \begin{bmatrix} 0 & 0 & 0 \\ -2 & 1 & 2 \\ -2 & 1 & 2 \end{bmatrix} =: A'$$

Since the first row is $[0\ 0\ 0]$, we are done.

b Need to find all v s.t. $Av = 3v$ or $(A - 3I)v = 0$.

From the theory of linear equations, we see that solutions of $A'v = 0$ are those of

$$\begin{bmatrix} \boxed{-2} & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} v = 0.$$

Basic solutions: • $x_2 = 1$, $x_3 = 0$

$$v_1 = \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}$$

• $x_2 = 0$, $x_3 = 1$

$$v_2 = \begin{bmatrix} 1 \\ 0 \\ \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Thus, the set of all eigenvectors is:

$$\left\{ \alpha \begin{bmatrix} 1/2 \\ 1 \\ 6 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} : \alpha, \beta \in \mathbb{R} \right\} \setminus \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

More concisely: $\left(\text{span}\{v_1, v_2\}\right) \setminus \{\mathbf{0}\}$.

2 $A \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 4 & 2 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$$= \begin{bmatrix} 0 \\ 6 \\ 6 \end{bmatrix} = 6 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Thus, $\begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T$ is an e-vec. with e-val 6.

(Note $\begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T \neq \mathbf{0}$.)

4.7

31 March 2021 10:58

4.7 Let $\theta \in (-\pi, \pi]$, $\mathbf{A} := \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and $\mathbb{K} = \mathbb{C}$. ① Show that $\cos\theta \pm i\sin\theta$ are eigenvalues of $\mathbf{A}$.
② Find an invertible matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{AP}$ is a diagonal matrix, and check your answer.

① Note

$$p_A(t) = \det(A - tI)$$

$$= \det \begin{bmatrix} \cos\theta - t & -\sin\theta \\ \sin\theta & \cos\theta - t \end{bmatrix}$$

$$= (\cos\theta - t)^2 + (\sin\theta)^2$$

$$= t^2 - (2\cos\theta)t + 1$$

Now, note that

$$p(\cos\theta + i\sin\theta) = (\cos\theta + i\sin\theta)^2 - (2\cos\theta)(\cos\theta + i\sin\theta) + 1$$

$$= [\cos^2\theta - \sin^2\theta] + \cancel{2i\sin\theta\cos\theta} - [2\cos^2\theta] - \cancel{2i\sin\theta\cos\theta} + 1$$

$$= -(\cos^2\theta + \sin^2\theta) + 1 = 0.$$

Similarly, $p(\cos\theta - i\sin\theta) = 0.$

Thus, $\cos\theta \pm i\sin\theta$ are eigenvalues.

②

Proposition

A matrix $\mathbf{A} \in \mathbb{K}^{n\times n}$ is diagonalizable if and only if there is a basis for $\mathbb{K}^{n\times 1}$ consisting of eigenvectors of $\mathbf{A}$. In fact,

$\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$, where $\mathbf{P}, \mathbf{D} \in \mathbb{K}^{n\times n}$ are of the form

$\mathbf{P} = \begin{bmatrix} \mathbf{x}_1 & \cdots & \mathbf{x}_n \end{bmatrix}$ and $\mathbf{D} = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$

(2)

Proposition

A matrix $\mathbf{A} \in \mathbb{K}^{n\times n}$ is diagonalizable if and only if there is a basis for $\mathbb{K}^{n\times 1}$ consisting of eigenvectors of $\mathbf{A}$. In fact,

$$\begin{aligned} & \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}, \text{ where } \mathbf{P}, \mathbf{D} \in \mathbb{K}^{n\times n} \text{ are of the form} \\ & \mathbf{P} = \begin{bmatrix} \mathbf{x}_1 & \cdots & \mathbf{x}_n \end{bmatrix} \text{ and } \mathbf{D} = \operatorname{diag}(\lambda_1, \ldots, \lambda_n) \\ \iff & \{\mathbf{x}_1, \ldots, \mathbf{x}_n\} \text{ is a basis for } \mathbb{K}^{n\times 1} \text{ and} \\ & \mathbf{A}\mathbf{x}_k = \lambda_k \mathbf{x}_k \text{ for } k = 1, \ldots, n. \end{aligned}$$

Let us find an e-vector corresponding to $e^{i\theta}$.

(Option 1: Use the theory of lin. equations as before.)

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Consider $x_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

$$Ax_1 = \begin{bmatrix} \cos\theta + i\sin\theta \\ \sin\theta - i\cos\theta \end{bmatrix} = (\cos\theta + i\sin\theta)\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Thus, x_1 is an evector corresp. to $e^{i\theta}$.

Similarly, $x_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}$ is an e-vector corresp. to $e^{-i\theta}$.

Thus, $P = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$ is one such matrix.

$$P^{-1} = \frac{1}{2i}\begin{bmatrix} i & -1 \\ i & 1 \end{bmatrix}.$$

Now, $P^{-1}AP = \dfrac{1}{2i}\begin{bmatrix} i & -1 \\ i & 1 \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$

$$= \frac{1}{2i}\begin{bmatrix} i & -1 \\ i & 1 \end{bmatrix}\begin{bmatrix} e^{i\theta} & e^{-i\theta} \\ -i\,e^{i\theta} & i\,e^{-i\theta} \end{bmatrix}$$

$$= \frac{1}{2i}\begin{bmatrix} 2i\,e^{i\theta} & 0 \\ 0 & 2i\,e^{-i\theta} \end{bmatrix} = \begin{bmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{bmatrix}$$

diagonal!

4.8

31 March 2021 13:15

4.8 Let $n \geq 2$ and $\mathbf{A} := \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \vdots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} \in \mathbb{R}^{n \times n}$, that is, $a_{jk} = 1$ for all $j, k = 1, \dots, n$. ① Find rank $\mathbf{A}$ and nullity $\mathbf{A}$. ② Find an eigenvector of $\mathbf{A}$ corresponding to a nonzero eigenvalue by inspection. ③ Find two distinct eigenvalues of $\mathbf{A}$ along with their geometric multiplicities, and find bases for the eigenspaces. ④ Show that $\mathbf{A}$ is diagonalizable, and find an invertible matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is a diagonal matrix.

① $$RCF(A) = \begin{bmatrix} 1 & \cdots & 1 \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$$

Thus, $\operatorname{rank}(A) = 1$ and $\operatorname{nullity}(A) \underset{\text{rank-nullity theorem}}{=} n-1 > 0$.

$\operatorname{nullity}(A) = \operatorname{nullity}(A - 0I)$.

Thus, 0 is an eigenvalue ($\because n \geq 2$) and its geo-mult. is $n-1$.

② Let $v = [1 \ \cdots \ 1]^T \in \mathbb{R}^{n \times 1}$. Then $Av = nv$ and $v \neq 0$.
Thus, n is an eigenvalue.

(Ex. Show that a 9x9 solved Sudoku has 45 as an eigenvalue)

③ We already have 0 and n as e-values.

Basis of $\mathcal{N}(A - 0I)$:
$$B = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} -1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right\}.$$

$w_1 \qquad w_2 \qquad w_{n-1}$

We know: $\dim(\mathcal{N}(A - nI)) \geq 1$.

Claim. $\dim(\mathcal{N}(A - nI)) = 1$.

Proof. Suppose not. Let $v_1, v_2 \in \mathcal{N}(A - nI)$ be lin. indep. and distinct.
Then, $Av_1 = nv_1$ and $Av_2 = nv_2$.
Thus, $v_1, v_2 \notin B$.

(Sub) Claim. $B' = B \cup \{v_1, v_2\}$ is lin. indep.

Proof. Suppose $\alpha_1, \dots, \alpha_{n+1}$ are s.t.

$$\alpha_1 w_1 + \cdots + \alpha_{n-1} w_{n-1} + \alpha_n v_1 + \alpha_{n+1} v_2 = 0. \quad (*)$$

Multiplying with $(A - nI)$ gives

$$-n(\alpha_1 w_1 + \cdots + \alpha_{n-1} w_{n-1}) + 0 + 0 = 0.$$

Since B is lin. indep., we get

$$\alpha_1 = \cdots = \alpha_{n-1} = 0.$$

Put back in $(*)$ and use lin. indep. of $\{v_1, v_2\}$ to get $\alpha_n = \alpha_{n+1} = 0$.
This proves the sub-claim. $\blacksquare$

But $|B'| = n+1$ and $B' \subset \mathbb{R}^{n \times 1}$. A contradiction.
This proves the claim. $\blacksquare$

Thus, geo-mult(n) $= 1$ and $\left\{ \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \right\}$ is a basis.

4 As before, $P = \begin{bmatrix} -1 & -1 & & -1 & 1 \\ 1 & 0 & & 0 & 1 \\ 0 & 1 & \cdots & 0 & 1 \\ 0 & 0 & & 0 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & & 0 & 1 \\ 0 & 0 & & 1 & 1 \end{bmatrix}$ works.

Questions for Thought

31 March 2021 10:57

1. Show that every <u>SOLVED</u> 9×9 Sudoku has 45 as eigenvalue.

2. Let $A \in \mathbb{K}^{2\times 2}$ have eigenvalues λ, μ.

(a) Show that if $\lambda \neq \mu$, then $A \sim \begin{bmatrix} \lambda & \\ & \mu \end{bmatrix}$.

(b) Show that if $\lambda = \mu$, then either $A \sim \begin{bmatrix} \lambda & \\ & \lambda \end{bmatrix}$ or $A \sim \begin{bmatrix} \lambda & 1 \\ & \lambda \end{bmatrix}$.

Show that $\begin{bmatrix} \lambda & \\ & \lambda \end{bmatrix} \not\sim \begin{bmatrix} \lambda & 1 \\ & \lambda \end{bmatrix}$.