2.1

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2.1 Find the Row Canonical Form of $\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 2 & 0 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 2 & 0 \end{bmatrix}$$

$R_3 \mapsto R_3 - R_1$

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 1 & -1 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \leftarrow \text{REF, not RCF}$$

$R_2 \mapsto (-1) R_2$

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$R_1 \mapsto R_1 - 2R_2$

$$\begin{bmatrix} 1 & 0 & 3 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$R_2 \mapsto R_2 + R_3, \quad R_1 \mapsto R_1 - 3R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \rightarrow \text{RCF} \checkmark$$

2.2

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2.2 Let $\mathbf{A} := \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. Find $\mathbf{A}^{-1}$ by Gauss-Jordan method.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \mapsto R_2 - R_1, \quad R_3 \mapsto R_3 - R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right]$$

$R_3 \mapsto R_3 - R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

$\uparrow$

I

$\therefore$ A is invertible and $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

2.3

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2.3 An $m \times m$ matrix $\mathbf{E}$ is called an **elementary matrix** if it is obtained from the identity matrix $\mathbf{I}$ by an elementary row operation. Write down all elementary matrices.

(i) Let $\mathbf{A} \in \mathbb{R}^{m \times n}$. If an elementary row operation transforms $\mathbf{A}$ to $\mathbf{A}'$, then show that $\mathbf{A}' = \mathbf{EA}$, where $\mathbf{E}$ is the corresponding elementary matrix.

(ii) Show that every elementary matrix is invertible, and find its inverse.

(iii) Show that a square matrix $\mathbf{A}$ is invertible if and only if it is a product of finitely many elementary matrices.

(0) All elem. matrices.

Type I : Interchange two rows.

$$E_{i,j} = \begin{array}{c c} & \begin{array}{cc} i^{th} & j^{th} \end{array} \\ \begin{array}{c} \\ i^{th} \\ \\ j^{th} \\ \\ \end{array} & \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 0 & \cdots & 1 \\ & & \vdots & \ddots & \vdots \\ & & 1 & \cdots & 0 \\ & & & & & \ddots \\ & & & & & & 1 \end{bmatrix} \end{array} \qquad (i<j)$$

$E_{i,j} = [e_{kl}]$ where

$$e_{kl} = \begin{cases} 1 & ; \quad k = l \neq i \text{ and } k = l \neq j \\ 1 & ; \quad k = j,\ l = i \text{ or } k = i,\ l = j \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Type II : Add a scalar multiple α of R_j to R_i. $(j \neq i)$

$$E_{i,j}(\alpha) = \begin{array}{c c} & \begin{array}{cc} i^{th} & j^{th} \text{ column} \end{array} \\ \begin{array}{c} \\ i^{th} \text{ row} \\ j^{th} \text{ row} \\ \end{array} & \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & \cdots & \alpha \\ & & & \ddots & \vdots \\ & & & & 1 \\ & & & & & \ddots & 1 \end{bmatrix} \end{array}$$

entries: $e_{kl} = \begin{cases} 1 ; & k = l \\ \alpha ; & k = i, \ l = j \\ 0 ; & \text{otherwise} \end{cases}$

Type III : Multiply a row with a <u>non-zero</u> scalar α.

$$E_i(\alpha) = \begin{array}{c} \\ i \end{array} \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \alpha & \\ & & & & 1 \\ & & & & & \ddots \\ & & & & & & 1 \end{bmatrix}$$

$$e_{kl} = \begin{cases} 1 ; & k = l \neq i \\ \alpha ; & k = l = i \\ 0 ; & o/w \end{cases}$$

(i). Note that $A = B$ iff A and B have the same rows (in same order)

Let $e_i = [0 \cdots 0 \ 1 \ 0 \cdots 0] \in \mathbb{R}^{1 \times m}$.
$\uparrow i^{th}$

Then, $e_i A$ is the i^{th} row of A.
Thus, $A = B \iff e_i A = e_i B \ \forall \ 1 \leq i \leq m$.

To show: $A' = EA$.
Suffices: $e_k A' = e_k E A \qquad \forall \ 1 \leq k \leq m$

- Type I: $E = E_{i,j}$ (interchange matrix)

$$\begin{aligned} e_i A' &= i^{th} \text{ row of } A' \\ &= j^{th} \text{ row of } A \\ &= e_j A \\ &= e_i E_{i,j} A = e_i E A \end{aligned}$$

Thus, $e_i A' = e_i EA$.

Similarly $e_j A' = e_j EA$. (By symmetry.)

Now, if $k \neq i, j$, then

$$\begin{aligned} e_k A' &= k^{th} \text{ row of } A' \\ &= k^{th} \text{ row of } A \\ &= e_k A = e_k E_{i,j} A. \end{aligned}$$

Thus, $e_k A' = e_k EA \quad \forall\ 1 \leq k \leq m$.

- Type II and III : <u>Exercise</u>.

(ii) To show: E is invertible.

Verify : <u>Type I</u>. E_{ij} is its own inverse.

<u>Type II</u>. $E_{i,j}(\alpha)$ is inverse of $E_{i,j}(-\alpha)$.

<u>Type III</u>. $E_i(\alpha)$ is inverse of $E_i(1/\alpha)$.
(Recall that $\alpha \neq 0$ in Type III.)

(iii) A is invertible $\Leftrightarrow$ clear

$\Leftrightarrow$ RCF of A is I $\Leftrightarrow$ clear

$\Leftrightarrow$ $\exists$ EROs converting A to I $\Leftrightarrow$ by earlier part

$\Leftrightarrow$ $\exists E_1, \dots, E_s$ s.t. $E_s \cdots E_1 A = I$

$$E_s \cdots E_1 A = I \Leftrightarrow A = E_1^{-1} \cdots E_s^{-1}$$

part (ii) shows that all these inverses are again elementary.

(of same type!)

2.4

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T and S could possibly be infinite.

2.4 Let S and T be subsets of $\mathbb{R}^{n\times 1}$ such that $S \subset T$. Show that if S is linearly dependent then so is T, and if T is linearly independent then so is S. Does the converse hold?

(i) S linearly dep $\Rightarrow$ T linearly dep.

Proof. Since S is lin. dep., $\exists\, v_1, \dots, v_s \in S$ and $\alpha_1, \dots, \alpha_s \in \mathbb{R}$ not all zero s.t.

$$\alpha_1 v_1 + \cdots + \alpha_s v_s = \mathbf{0}.$$

Since $S \subset T$, each $v_i \in T$.
Thus, the above shows that T is lin. dep.

(ii) T is linearly independent $\Rightarrow$ S is linearly independent

Proof. (ii) is the contrapositive of (i). ∎

Statement (I): $P \Rightarrow Q$

Contrapositive (II): $\neg Q \Rightarrow \neg P$

(I) is true $\Leftrightarrow$ (II) is true

(iii) Is converse true?

Converse: S independent $\Rightarrow$ T independent

Ans. No. (Counter) Example ① $S = \phi$

$T = \{\mathbf{0}\} \subseteq \mathbb{R}^{n\times 1}$.

② $S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\},$

2. $S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$,

$T = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

All proper subsets are lin. indep

2.5

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2.5 Are the following sets linearly independent?

(i) $\{[1 \;\; -1 \;\; 1], [3 \;\; 5 \;\; 2], [1 \;\; 2 \;\; 1], [1 \;\; 1 \;\; 1]\} \subset \mathbb{R}^{1\times 3}$,

(ii) $\{[1 \;\; 9 \;\; 9 \;\; 8], [2 \;\; 0 \;\; 0 \;\; 3], [2 \;\; 0 \;\; 0 \;\; 8]\} \subset \mathbb{R}^{1\times 4}$,

(iii) $\{[1 \;\; -1 \;\; 0]^\mathsf{T}, [3 \;\; -5 \;\; 2]^\mathsf{T}, [1 \;\; -2 \;\; 1]^\mathsf{T}\} \subset \mathbb{R}^{3\times 1}$.

(i) No. If we have m vectors in $\mathbb{R}^{1\times n}$ with $m > n$, then they are lin. dep.
Here, $m = 4$, $n = 3$.

(ii) Put them in a matrix with the vectors as columns.

$$\begin{bmatrix} 1 & 2 & 2 \\ 9 & 0 & 0 \\ 9 & 0 & 0 \\ 8 & 3 & 5 \end{bmatrix}$$

$R_3 \mapsto R_3 - R_2$

$$\begin{bmatrix} 1 & 2 & 2 \\ 9 & 0 & 0 \\ 0 & 0 & 0 \\ 8 & 3 & 5 \end{bmatrix}$$

$R_3 \leftrightarrow R_4$

$$\begin{bmatrix} 1 & 2 & 2 \\ 9 & 0 & 0 \\ 8 & 3 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_2 \mapsto R_2 - 9R_1$$
$$R_3 \mapsto R_3 - 8R_1$$

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & -18 & -18 \\ 0 & -13 & -11 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_3 \mapsto R_3 - \left(\frac{13}{18}\right)R_2$$

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & -18 & -18 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

row rank $= 3 = \#$ of vectors.
Thus, the vectors are linearly independent.

(iii) $$\begin{bmatrix} 1 & 3 & 1 \\ -1 & -5 & -2 \\ 0 & 2 & 1 \end{bmatrix}$$

$$R_2 \mapsto R_2 + R_1$$

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & -2 & -1 \\ 0 & 2 & 1 \end{bmatrix}$$

$$R_3 \mapsto R_3 + R_2$$

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, row-rank $= 2 <$ # of vectors. $\therefore$ DEPendent.

2.6

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S

2.6 Given a set of s linearly independent row vectors $\{\mathbf{a}_1, \dots, \mathbf{a}_i, \dots, \mathbf{a}_j, \dots, \mathbf{a}_s\}$ in $\mathbb{R}^{1\times n}$ and $\alpha \in \mathbb{R}$, show that the set $\{\mathbf{a}_1, \dots, \mathbf{a}_{i-1}, \mathbf{a}_i + \alpha \mathbf{a}_j, \mathbf{a}_{i+1}, \dots, \mathbf{a}_j, \dots, \mathbf{a}_s\}$ is linearly independent.

$i \neq j$ is an assumption

Suppose $\alpha_1, \alpha_2, \dots, \alpha_s \in \mathbb{R}$ are such that

$$\alpha_1 a_1 + \cdots + \alpha_{i-1} a_{i-1} + \alpha_i (a_i + \alpha a_j) + \alpha_{i+1} a_{i+1} + \cdots + \underset{\alpha_j a_j}{\cdots} + \alpha_s a_s = \mathbf{0}$$

Want: To show that each $\alpha_k = 0$.
(That is, it is forced $\alpha_k = 0$)

$$\alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_{i-1} a_{i-1} + \alpha_i a_i + \alpha_{i+1} a_{i+1} + \cdots + (\alpha_i \alpha + \alpha_j) a_j + \cdots + \alpha_s a_s = \mathbf{0}.$$

Since S is linearly independent,

$$\alpha_1 = \alpha_2 = \cdots = \alpha_{j-1} = \alpha_i \alpha + \alpha_j = \alpha_{j+1} = \cdots = \alpha_s = 0.$$

Thus, $\alpha_k = 0$ for all $k \neq j$.
Since $\alpha_i \alpha + \alpha_j = 0$ and $\alpha_i = 0$, we get $\alpha_j = 0$ as well.

Thus, $\alpha_k = 0$ for all k, as desired.

2.7

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2.7 Find the ranks of the following matrices.

$$\text{(i)} \begin{bmatrix} 8 & -4 \\ -2 & 1 \\ 6 & -3 \end{bmatrix}, \quad \text{(ii)} \begin{bmatrix} 0 & 8 & -1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 4 & 5 \end{bmatrix}.$$

Recall: ① EROs don't change row-rank.
② row-rank of REF is # non-zero rows

(i) $\begin{bmatrix} 8 & -4 \\ -2 & 1 \\ 6 & -3 \end{bmatrix}$

$R_2 \mapsto R_2 + \frac{1}{4} R_1$
$R_3 \mapsto R_3 - \frac{6}{8} R_1$

$$\begin{bmatrix} 8 & -4 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Thus, rank $= 1$.

(ii) $\begin{bmatrix} 0 & 8 & -1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 4 & 5 \end{bmatrix}$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 8 & -1 \\ 0 & 0 & 3 \\ 0 & 4 & 5 \end{bmatrix}$$

$R_4 \mapsto R_4 - \frac{1}{2} R_2$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 8 & -1 \\ 0 & 0 & 3 \\ 0 & 0 & 11/2 \end{bmatrix}$$

$$R_4 \mapsto R_4 - \left(\frac{11/2}{3}\right) R_3$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 8 & -1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \text{REF}$$

$\therefore$ row-rank $= \underline{\underline{3}}$.