1.1

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$$\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 1$$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \neq I_4$$

1.1 Let $\mathbf{A}$ be a square matrix. Show that there is a symmetric matrix $\mathbf{B}$ and there is a skew-symmetric matrix $\mathbf{C}$ such that $\mathbf{A} = \mathbf{B} + \mathbf{C}$. Are $\mathbf{B}$ and $\mathbf{C}$ unique?

Sol

Symmetric : $B^T = B$

Skew-symmetric : $C^T = -C$

Note : Anything in purple is for the sake of discussion.

Idea: $(A = B + C)$ ← Not given but what I want.

$A^T = B^T + C^T = B - C$

$$B = \frac{A + A^T}{2}, \qquad C = \frac{A - A^T}{2}$$

Consider $B = \frac{1}{2}(A + A^T) \in \mathbb{R}^{n \times n}$ and

$C = \frac{1}{2}(A - A^T) \in \mathbb{R}^{n \times n}$.

Observe,
$$B^T = \left(\frac{1}{2}(A + A^T)\right)^T = \frac{1}{2}(A + A^T)^T$$
$$= \frac{1}{2}(A^T + (A^T)^T)$$
$$= \frac{1}{2}(A^T + A) = \frac{1}{2}(A + A^T) = B.$$

Similarly,
$$C^T = \left(\frac{1}{2}(A - A^T)\right)^T = \frac{1}{2}(A^T - (A^T)^T)$$
$$= \frac{1}{2}(A^T - A) = -\frac{1}{2}(A - A^T) = -C.$$

Thus, B is sym. and C is skew-symm.

Moreover $B + C = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$

$$= A.$$

Thus, we have shown existence.

Uniqueness: Suppose B', C' are sym. & skew-sym., resp. such that

$$A = B' + C' \qquad \text{— (1)}$$

Want to show: $B = B'$ and $C = C'$.

Now, B' is sym. and C' skew-sym, we see

$$A^T = (B')^T + (C')^T = B' - C'. \qquad \text{— (2)}$$

Solving (1) and (2) gives

$$B' = \frac{1}{2}(A + A^T) = B \text{ and}$$

$$C' = \frac{1}{2}(A - A^T) = C.$$

1.2

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1.2 Let $\mathbf{A} := \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ and $\mathbf{B} := \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. Write (i) the second row of $\mathbf{AB}$ as a linear combination of the rows of $\mathbf{B}$ and (ii) the second column of $\mathbf{AB}$ as a linear combination of the columns of $\mathbf{A}$.

(i) $$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} * & * & * \\ 3\cdot 1 + 4\cdot 4 & 3\cdot 2 + 4\cdot 5 & 3\cdot 3 + 4\cdot 6 \\ * & * & * \end{bmatrix}$$

Thus, the second row is:

$$3 \cdot [1 \quad 2 \quad 3] + 4 \cdot [4 \quad 5 \quad 6].$$

rows of B

(ii) $$AB = \begin{bmatrix} * & 1\cdot 2 + 2\cdot 5 & * \\ * & 3\cdot 2 + 4\cdot 5 & * \\ * & 5\cdot 2 + 6\cdot 5 & * \end{bmatrix}$$

Second column is: $$2 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + 5 \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}.$$

1.3

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1.3 Let $\mathbf{A} := \begin{bmatrix} 1 & 1 & 1 & 0 \\ -3 & -17 & 1 & 2 \\ 4 & -24 & 8 & -5 \\ 0 & -7 & 2 & 2 \end{bmatrix}$. Assuming that $\mathbf{A}$ is invertible, find the last column and the last row of $\mathbf{A}^{-1}$.

General. Let $B = A^{-1} \in \mathbb{R}^{n \times n}$.

Let $B = [b_1 \; \cdots \; b_n]$ for $b_1, \dots, b_n \in \mathbb{R}^{n \times 1}$.

Then $AB = [Ab_1 \; \cdots \; Ab_n]$ but also $AB = I_n$.

Thus, $Ab_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$

In our case $n = 4$. Let $b_n = [x_1 \; x_2 \; x_3 \; x_4]^T$.

We wish to solve $\begin{bmatrix} 1 & 1 & 1 & 0 \\ -3 & -17 & 1 & 2 \\ 4 & -24 & 8 & -5 \\ 0 & -7 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

We convert to REF.

- $R_2 \mapsto R_2 + 3R_1, \quad R_3 \mapsto R_3 - 4R_1$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & -14 & 4 & 2 & 0 \\ 0 & -28 & 4 & -5 & 0 \\ 0 & -7 & 2 & 2 & 1 \end{array}\right]$$

- $R_3 \mapsto R_3 - 2R_2 \; ; \; R_4 \mapsto R_4 - R_2/2$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & -14 & 4 & 2 & 0 \\ 0 & 0 & -4 & -9 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

This gives $x_4 = 1$

$-4x_3 - 9x_4 = 0$ or $x_3 = -\frac{9}{4}$

$-14x_2 + 4x_3 + 2x_4 = 0$ or $x_2 = \frac{1}{7}(2x_3 + x_4)$

$= \frac{1}{7}(1 - 9/2) = -\frac{1}{2}$

$x_1 = -(x_2 + x_3)$ or $x_1 = \frac{11}{4}$.

To get the last row, we write

$$B = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} \quad \text{for} \quad c_1, \dots, c_4 \in \mathbb{R}^{1\times n}.$$

We use $BA = \begin{bmatrix} c_1 A \\ \vdots \\ c_4 A \end{bmatrix}$ to get

$$c_4 A = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$$

or

$$A^T c_4^T = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

We are in a similiar position as earlier.

Get $c_4 = \begin{bmatrix} -3/2 & -1/2 & 0 & 1 \end{bmatrix}$

1.4

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1.4 Show that the product of two upper triangular matrices is upper triangular. Is this true for lower triangular matrices?

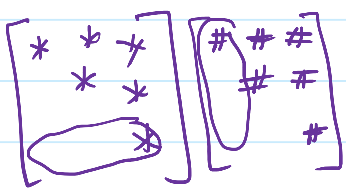
- Let $A, B \in \mathbb{R}^{n \times n}$ be upper triangular,
  where $A = [a_{ij}]$, $B = [b_{ij}]$.

  We have $a_{ij} = b_{ij} = 0$ when $i > j$.
  $(i, j \in \{1, \dots, n\})$

  Let $P = AB = [p_{ij}]$.
  We wish to show: P is upper triangular,
  i.e., $p_{ij} = 0 \quad \forall i > j$.
  $(i, j \in \{1, \dots, n\})$

  Let $1 \le j < i \le n$ be arbitrary.
  We have

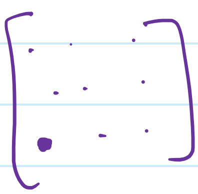
$$p_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$



$$= \sum_{k=1}^{i-1} a_{ik} b_{kj} + \sum_{k=i}^{n} a_{ik} b_{kj}$$

$a_{ik} = 0$ since $k \le i-1 < i$

$b_{kj} = 0$ since $k \ge i > j$



$$= \sum_{k=1}^{i-1} 0 \cdot b_{kj} + \sum_{k=i}^{n} a_{ik} \cdot 0$$

$$= 0.$$

  Since i, j were arbitrary, we are done.

- For lower triangle: ① Imitate the proof.
  or ② Use that

A is lower triang $\Leftrightarrow$ A^T is upper tri.

$A, B \rightarrow$ lower triangular

$\Downarrow$

A^T, B^T upper triangular $\xRightarrow{\text{previous part}}$ $B^T A^T$ is upper trian

$\Downarrow$

$(B^T A^T)^T$ is lower trian.

$\parallel$

AB

1.5

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Is trace(ABC) = trace(ACB) ? No. Find example.

trace(ABC) = trace(BCA) = trace(CAB) ✓

1.5 The **trace** of a square matrix is the sum of its diagonal entries. Show that trace $(\mathbf{A}+\mathbf{B}) = $ trace $(\mathbf{A})+$ trace $(\mathbf{B})$ and trace $(\mathbf{AB}) = $ trace $(\mathbf{BA})$ for $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$.

- Sum. Let $S = A + B$. $\left(\begin{array}{l} A = [a_{ij}] \\ B = [b_{ij}] \end{array} \right)$

Write $S = [s_{ij}]$.

$$s_{ij} = a_{ij} + b_{ij} \quad \forall \ 1 \le i, j \le n$$

$$\Rightarrow \text{trace } S = \sum_{i=1}^{n} s_{ii} \qquad \text{(by def}^{n}\text{)}$$

$$= \sum_{i=1}^{n} (a_{ii} + b_{ii})$$

$$= \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii}$$

$$= \text{trace } A + \text{trace } B \qquad \text{(by def}^{n}\text{)}$$

- trace (AB) = trace (BA)

$C = AB$, $D = BA$. $C = [c_{ij}]$, $D = [d_{ij}]$.

$$\text{trace } C = \sum_{i=1}^{n} c_{ii}$$

$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} b_{ji} \right)$$

$$\text{trace } D = \sum_{j=1}^{n} d_{jj}$$

$$= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} b_{ji} a_{ij} \right)$$

$$= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} a_{ij} b_{ji} \right)$$

$$= \sum_{j=1} \left(\sum_{i=1} a_{ij} b_{ji} \right)$$

$$= \sum_{1 \le i,j \le n} a_{ij} b_{ji}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{ji} = \text{trace } C. \qquad \blacksquare$$

1.6

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1.6 Find all solutions of the linear system $\mathbf{Ax} = \mathbf{b}$, where (i) $\mathbf{A} := \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix}$, $\mathbf{b} := \begin{bmatrix} 0 & -1 & 6 & 6 \end{bmatrix}^\mathsf{T}$,

(ii) $\mathbf{A} := \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$, $\mathbf{b} := \begin{bmatrix} 5 & -2 & 9 \end{bmatrix}^\mathsf{T}$,

(iii) $\mathbf{A} := \begin{bmatrix} 0 & 2 & -2 & 1 \\ 2 & -8 & 14 & -5 \\ 1 & 3 & 0 & 1 \end{bmatrix}$ and $\mathbf{b} := \begin{bmatrix} 2 & 2 & 8 \end{bmatrix}^\mathsf{T}$

by reducing $\mathbf{A}$ to a row echelon form.

(i) $$[A|b] = \left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 6 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{array}\right]$$

Want: REF. ① $R_2 \mapsto R_2 - 2R_1$, $R_4 \mapsto R_4 - 2R_1$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 6 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right]$$

② $R_3 \mapsto R_3 + 5R_2$, $R_4 \mapsto R_4 + 4R_2$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{array}\right]$$

③ $R_3 \leftrightarrow R_4$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \\ (0 & 0 & 0 & 0 & 0 & 0) & 1 \end{array}\right]$$

zero   non-zero

Thus, the system has no solution, i.e., the system is inconsistent.

(ii) $[A|b]$ $\left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{array}\right]$

① $R_2 \mapsto R_2 - 2R_1$, $\quad R_3 \mapsto R_3 + R_1$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 8 & 3 & 14 \end{array}\right]$$

② $R_3 \mapsto R_3 + R_2$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Thus, we have a consistent (has at least one solution). How many?

Exactly one. There are no free variables. (Every column has a pivot.)

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -12 \\ 2 \end{bmatrix}$$

Back-substitute and retrieve.

$$x_3 = \frac{2}{1} = 2.$$

$$-8x_2 - 2x_3 = -12 \rightsquigarrow \text{get } x_2$$

$$2x_1 + x_2 + x_2 = 5 \rightsquigarrow \text{get } x_1$$

(iii) $$\left[\begin{array}{cccc|c} 0 & 2 & -2 & 1 & 2 \\ 2 & -8 & 14 & -5 & 2 \\ 1 & 3 & 0 & 1 & 8 \end{array}\right]$$

- $R_1 \leftrightarrow R_3$

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 1 & 8 \\ 2 & -8 & 14 & -5 & 2 \\ 0 & 2 & -2 & 1 & 2 \end{array}\right]$$

- $R_2 \mapsto R_2 - 2R_1$

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 1 & 8 \\ 0 & -14 & 14 & -7 & -14 \\ 0 & 2 & -2 & 1 & 2 \end{array}\right]$$

- $R_3 \mapsto R_3 + \frac{1}{7} R_2$

$$\left[\begin{array}{cccc|c} \boxed{1} & 3 & 0 & 1 & 8 \\ 0 & \boxed{-14} & 14 & -7 & -14 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

System is consistent!

2 free variables: x_3 and x_4.

- Particular solution. Put $x_3 = x_4 = 0$ to get x_1 and x_2.

$$0 - 14x_2 + 14\cancel{x_3}^{\,0} + (-7)\cancel{x_4}^{\,0} = -14$$

$$\Rightarrow x_2 = 1$$

$$x_1 + 3x_2 + 0 + 0 = 8 \Rightarrow x_1 = 5$$

Thus, $\begin{bmatrix} 5 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ is a particular solution.

- General solution to $Ax = 0$.

(i) $x_3 = 1$, $x_4 = 0$

Get x_1 and x_2 by back-substitution.

$$\begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

(ii) $x_3 = 0$, $x_4 = 1$.

Get x_1 and x_2. We get

$$\begin{bmatrix} 1/2 \\ -1/2 \\ 0 \\ 1 \end{bmatrix}.$$

- Thus, general solution of $Ax = b$ is given as

$$\left\{ \begin{bmatrix} 5 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \\ 1 \end{bmatrix} : \alpha, \beta \in \mathbb{R} \right\}^{\in \mathbb{R}^{4\times 1}}.$$

Questions For Thought

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Q1. When is an upper triangular matrix invertible?
Can you prove this?

Q2. Assume that U is an invertible upper Δ matrix.
Show that U^{-1} is also upper Δ.

(Don't use any determinant / adjoint machinery.)

Q3. Let $f: X \to Y$ and $g: Y \to X$ be functions
such that $(f \circ g)(y) = y \quad \forall y \in Y$.
Does it follow that $(g \circ f)(x) = x \quad \forall x \in X$?

Q4. Is $\operatorname{trace}(ABC) = \operatorname{trace}(ACB)$?
Is $\operatorname{trace}(ABC) = \operatorname{trace}(CAB)$?

Q5. Let $A, B, P \in \mathbb{R}^{n \times n}$ be s.t. P is invertible and

$$A = P^{-1} B P.$$

Show that $\operatorname{trace} A = \operatorname{trace} B$.