(Extra)^2 Questions for MA 106

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Semester: Spring 2021
Latest update: 2021-03-29 21:50:29+05:30

These are questions that came out of some discussions.

Notations

1. $\mathbb{F}$ denotes an arbitrary field. You may read this to get an introduction to fields. Or may assume that $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. (Although your answers then may not work for a general field.)

2. Given a linear transformation T, $\mathcal{N}(T)$ denotes the null space of T.

1. A **nonempty** subset $J \subset \mathbb{F}^{n \times n}$ is said to be a *two-sided ideal* if it has the following properties:

   (a) (Closed under addition) For all $A, B \in J$, we have $A + B \in J$,

   (b) (Absorption) For all $A \in J$ and $C \in \mathbb{F}^{n \times n}$, we have $AC, CA \in J$.

   Show that the (two-sided) ideals of $\mathbb{F}^{n \times n}$ are precisely $\{O\}$ and $\mathbb{F}^{n \times n}$.

2. Let $A \in \mathbb{F}^{n \times n}$ be such that $Ay = y$ for all $y \in \mathbb{F}^{n \times 1}$. Show that $A = I$.
   HIDDEN: Consider $y = e_k$ for $k \in \{1, \ldots, n\}$.

3. Suppose $A \in \mathbb{R}^{2 \times 2}$ is such that $x^\top Ax = 0$ for all $x \in \mathbb{R}^{2 \times 1}$. Is it necessary that $A = O$?
   HIDDEN: No. Interpret $x^\top Ax$ as $\langle Ax, x \rangle$.

4. Let $P \in \mathbb{R}^{n \times n}$ be invertible and let $A = P^\top P$.
   Show that if $x \in \mathbb{R}^{n \times 1}$, then $x^\top Ax = 0 \iff x = 0$.

5. Let $A \in \mathbb{F}^{n \times n}$ be arbitrary. Show that

   (a) A can be written as a sum of two invertible matrices, and

   (b) A can be written as a sum of two non-invertible matrices.

6. Can every matrix $A \in \mathbb{F}^{n \times n}$ be written as a product LU where $L, U \in \mathbb{F}^{n \times n}$ are lower and upper triangular, respectively?
   HIDDEN: No. Try to find a counterexample for $n = 2$.

7. Let V and W be finite dimensional vector spaces over $\mathbb{F}$. Let $T : V \to W$ be a linear transformation and let $B = \{v_1, \ldots, v_k\}$ be a basis of $\mathcal{N}(T)$. Extend B to a basis $B' = B \cup \{v_{k+1}, \ldots, v_n\}$ of V. Show that $\{T(v_{k+1}), \ldots, T(v_n)\}$ is a basis of range (or image) of T. What can you say about the dimensions involved? Does this seem familiar?

8. Let V and W be finite dimensional vector spaces over $\mathbb{F}$. Let $T : V \to W$ be a linear transformation and let $B = (v_1, \ldots, v_k)$ be an ordered basis of $\mathcal{N}(T)$. Extend B to an ordered basis $B' = (v_1, \ldots, v_k, v_{k+1}, \ldots, v_n)$ of V. Fix any ordered basis C of W. What can you say about the first k columns of $M_{B'}^{C}(T)$? The last $n - k$?

9. Let V be a finite dimensional vector spaces over $\mathbb{F}$ and $T : V \to V$ a linear transformation. Show that there exist ordered bases B and C of T such that the matrix $M_B^C(T)$ is diagonal with entries $1, \ldots, 1, 0, \ldots, 0$.